Weighted Clones

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what is this talk about?

- clones $\leftrightarrow$ relational clones
  - why study clones: almost whole UA + fun
  - why study relational clones: CSP
  - why we care about $\leftrightarrow$: UA $\cup$ CSP, extensive use in UA (congruences, description of clones, ...)

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- weighted clones $\leftrightarrow$ weighted relational clones
  - why study weighted clones: ? + more fun (more math)
  - why study weighted relational clones: valued CSP
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- this talk:
  - what is weighted (relational) clone
  - what is known + open problems

clones, relational clones, and CSP

- Notation
  - D ... finite set (the domain)
  - $\mathbf{A}$ ... set of operations on D
  - $\mathbb{A}$ ... set of relations on D

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  - **Example:** if $R_1, R_2 \in \mathbb{A}$ then S defined by
    $S(x, y)$ iff $(\exists z)\ R_1(x, z) \wedge R_2(z, y, y)$ is in $\mathbb{A}$

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- **Def:** CSP over $\mathbb{A}$ is the problem to decide whether a pp-sentence (over $\mathbb{A}$) is true
  - **Example:** Is $(\exists x, y, z)\ R_1(x, z) \wedge R_2(z, y, y)$ true?
  - Complexity does not change if we add pp-definable relation
  - $\Rightarrow$ Complexity depends only on the relational clone of $\mathbb{A}$.

clones $\leftrightarrow$ relational clones

- clones and rel. clones are closed objects in a Galois correspondence given by:
- Def: $f : D^n \to D$ is compatible with $R \subseteq D^m$ if $\mathbf{d}_1, \ldots, \mathbf{d}_n \in R \Rightarrow f(\mathbf{d}_1, \ldots, \mathbf{d}_n) \in R$

$$
\begin{array}{rccccl}
 & f & f & \ldots & f & \\
 & \downarrow & \downarrow & \ldots & \downarrow & \\
\mathbf{d}_1 = & (d_{11}, & d_{12}, & \ldots, & d_{1m}) & \in R \\
\mathbf{d}_2 = & (d_{21}, & d_{22}, & \ldots, & d_{2m}) & \in R \\
\vdots & & & \vdots & & \\
\mathbf{d}_n = & (d_{n1}, & d_{n2}, & \ldots, & d_{nm}) & \in R \\
\hline
 & & & & & \Downarrow \\
f(\mathbf{d}_1, \ldots, \mathbf{d}_n) = & (b_1, & b_2, & \ldots, & b_m) & \in R
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- $\mathrm{Pol}(\mathbb{A})$ . . . all operations compatible with every $R \in \mathbb{A}$
  Fact: always a clone
- $\mathrm{Inv}(\mathbf{A})$ . . . all relations compatible with every $f \in \mathbf{A}$
  Fact: always a relational clone

- **Theorem:** Pol and Inv are mutually inverse bijections

  clones $\leftrightarrow$ relational clones

  Geiger; Bodnarčuk, Kalužnin, Kotov, Romov

proof for the algebraic side

Theorem (Geiger; Bodnarčuk, Kalužnin, Kotov, Romov)

If $\mathbf{A}$ *is a clone, then* $\mathbf{A} = \mathrm{Pol}(\mathrm{Inv}(\mathbf{A}))$.

Proof.

□

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assume $t \notin \mathbf{A}$ (say n-ary).

want: relation R in $\mathrm{Inv}(\mathbf{A})$ which is not compatible with t

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- R is compatible with every $f \in \mathbf{A}$ since
  $f(\mathbf{g}_1, \ldots, \mathbf{g}_m)$ (component-wise application of f)
  $= f(g_1, \ldots, g_m)$ (superposition)

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  $f(\mathbf{g}_1, \ldots, \mathbf{g}_m)$ (component-wise application of f)
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- t is not compatible with R since
  $t(\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_n) = \mathbf{t} \notin R$

□

vCSP, weighted relational clones, weighted clones

Cohen, Cooper, Creed, Jeavons, Živný

- relation $\to$ weighted relation
- CSP $\to$ vCSP
- relational clone $\to$ weighted relational clone
- operation $\to$ fractional operation, weighting (2 versions)
- clone $\to$ weighted clone

relations in a weird way

- Relation $R \subseteq D^n$ can be alternatively defined as a mapping $\rho : D^n \to \{0, \infty\}$ (or to $\{c, \infty\}$)
  - $\rho(\mathbf{d}) = 0$ if $\mathbf{d} \in R$ (no penalty for using $\mathbf{d}$)
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- pp-definitions $\approx$ minimizing a sum
  - **Example:** $S(x, y)$ iff $(\exists z)\ R_1(x, z) \wedge R_2(z, y, y)$
  - corresponds to $\sigma(x, y) = \min_z\ \rho_1(x, z) + \rho_2(z, y, y)$

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- CSP $\approx$ minimizing a sum (over all variables)
  - **Example:** Is $(\exists x, y, z)\ R_1(x, z) \wedge R_2(z, y, y)$ true?
  - corresponds to Find $\min_{x,y,z}\ \rho_1(x, z) + \rho_2(z, y, y)$.

weighted relation, vCSP

- **Def:** weighted relation is a mapping $\rho : D^n \to \overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$
  - $\rho(\mathbf{d}) = 0$ (low penalty for using $\mathbf{d}$)
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- **Def:** $\text{Feas}(\rho) = \{\mathbf{d} : \rho(\mathbf{d}) < \infty\} \subseteq D^n$
- $\mathbb{W}$ ... set of weighted relations
- **Def:** vCSP over $\mathbb{W}$ is the problem to minimize a sum (which uses only weighted relations from $\mathbb{W}$)
  - **Example:** Find $\min_{x,y,z} \ \rho_1(x, z) + \rho_2(z, y, y)$.
  - Complexity does not change if we add ... (next slide)
  - $\Rightarrow$ complexity only depends on weighted relational clone of $\mathbb{W}$

weighted relational clone

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  - is closed under addition of constant and non-negative scaling
    Example: if $\rho \in \mathbb{W}$ then $2\rho + 3 \in \mathbb{W}$

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  - contains the equality relation
  - is closed under addition of constant and non-negative scaling
    Example: if $\rho \in \mathbb{W}$ then $2\rho + 3 \in \mathbb{W}$
  - is closed under addition and minimization over some coordinates
    Example: if $\rho_1, \rho_2 \in \mathbb{W}$ then σ defined by
    $\sigma(x, y) = \min_z \ \rho_1(x, z) + \rho_2(z, y, y)$ is in $\mathbb{W}$

fractional operation

- **Def:** n-ary fractional operation ϕ is
  a probability distribution over n-ary operations,
  written as a formal linear combination of operations

$$\phi = \sum_{f:D^n \to D} \phi(f) f, \quad \text{where } 0 \leq \phi(f) \in \mathbb{Q}, \ \sum_f \phi(f) = 1$$

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  $\phi = 0.1\pi_1 + 0.4 \min + 0.5 \max \qquad \mathrm{supp}(\phi) = \{\pi_1, \min, \max\}$

- **Natural example:**

  $\phi = 0.5 \min + 0.5 \max \qquad \mathrm{supp}(\phi) = \{\min, \max\}$

- **Def:** Support of ϕ is $\mathrm{supp}(\phi) = \{f : \phi(f) > 0\}$

compatibility

- **Def:** n-ary $\phi = \sum \phi(f) f$ is compatible with $\rho : D^m \to \overline{\mathbb{Q}}$ if for any $\mathbf{d}_1, \ldots, \mathbf{d}_n \in D^m$

$$\mathrm{EXP}_{f \sim \phi} \quad \rho(\, f(\mathbf{d}_1, \ldots, \mathbf{d}_n)\,) \leq \mathrm{avg}\ \{\rho(\mathbf{d}_1), \ldots, \rho(\mathbf{d}_n)\}$$

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- equivalently

$$\sum_{f \in \mathrm{supp}(\phi)} \phi(f)\, \rho(\, f(\mathbf{d}_1, \ldots, \mathbf{d}_n)\,) \leq \frac{1}{n} \rho(\mathbf{d}_1) + \cdots + \frac{1}{n} \rho(\mathbf{d}_n)$$

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- **Example:** $D = \{0, 1\}$, $\phi = 0.5 \min + 0.5 \max$

$$0.5\, \rho(\max(\mathbf{d}_1, \mathbf{d}_2)) + 0.5\, \rho(\min(\mathbf{d}_1, \mathbf{d}_2)) \leq 0.5\, \rho(\mathbf{d}_1) + 0.5\, \rho(\mathbf{d}_2)$$

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- **Remark (submodularity):** $D^m \approx$ power set of $\{1, \ldots, m\}$

$$0.5\, \rho(\mathbf{d}_1 \cup \mathbf{d}_2) + 0.5\, \rho(\mathbf{d}_1 \cap \mathbf{d}_2) \le 0.5\, \rho(\mathbf{d}_1) + 0.5\, \rho(\mathbf{d}_2)$$

good news, bad news

- **good news:** if ϕ is compatible with every $\rho \in \mathbb{W}$, then ϕ is compatible with every $\rho \in \mathrm{wRelClo}(\mathbb{W})$

good news, bad news

- **good news:** if ϕ is compatible with every $\rho \in \mathbb{W}$, then ϕ is compatible with every $\rho \in \mathrm{wRelClo}(\mathbb{W})$
- **good news:** definition of compatibility is beautiful

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- **good news:** if ϕ is compatible with every $\rho \in \mathbb{W}$, then ϕ is compatible with every $\rho \in \mathrm{wRelClo}(\mathbb{W})$
- **bad news:** superposition (defined naturally) does not work

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  ϕ is compatible with every $\rho \in \mathrm{wRelClo}(\mathbb{W})$
- **bad news:** superposition (defined naturally) does not work
- recall for $\phi = 0.5\,\mathsf{min} + 0.5\,\mathsf{max}$ the inequality was
  $0.5\,\rho(\max(\mathbf{d}_1, \mathbf{d}_2)) + 0.5\,\rho(\min(\mathbf{d}_1, \mathbf{d}_2)) \leq 0.5\,\rho(\mathbf{d}_1) + 0.5\,\rho(\mathbf{d}_2)$

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- this is equivalent to

  $0.5\,\rho(\mathsf{max}(\mathbf{d}_1, \mathbf{d}_2)) + 0.5\,\rho(\mathsf{min}(\mathbf{d}_1, \mathbf{d}_2)) - 0.5\,\rho(\pi_1(\mathbf{d}_1, \mathbf{d}_2)) - 0.5\,\rho(\pi_2(\mathbf{d}_1, \mathbf{d}_2)) \leq 0$

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- **solution:** work with $\phi' = 0.5 \min + 0.5 \max - 0.5\pi_1 - 0.5\pi_2$ and define compatibility with RHS=0

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- **solution:** work with $\phi' = 0.5\min + 0.5\max - 0.5\pi_1 - 0.5\pi_2$ and define compatibility with RHS=0
- in general $\phi' = \phi - 1/n \sum_i \pi_i$

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- in general $\phi' = \phi - 1/n \sum_i \pi_i$
- sum of weights is 0 and

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- **good news:** if ϕ is compatible with every $\rho \in \mathbb{W}$, then ϕ is compatible with every $\rho \in \mathrm{wRelClo}(\mathbb{W})$
- **bad news:** superposition (defined naturally) does not work
- recall for $\phi = 0.5 \min + 0.5 \max$ the inequality was
  $0.5\,\rho(\max(\mathbf{d}_1, \mathbf{d}_2)) + 0.5\,\rho(\min(\mathbf{d}_1, \mathbf{d}_2)) \leq 0.5\,\rho(\mathbf{d}_1) + 0.5\,\rho(\mathbf{d}_2)$
- this is equivalent to
  $0.5\,\rho(\max(\mathbf{d}_1, \mathbf{d}_2)) + 0.5\,\rho(\min(\mathbf{d}_1, \mathbf{d}_2)) - 0.5\,\rho(\pi_1(\mathbf{d}_1, \mathbf{d}_2)) - 0.5\,\rho(\pi_2(\mathbf{d}_1, \mathbf{d}_2)) \leq 0$
- **solution:** work with $\phi' = 0.5 \min + 0.5 \max - 0.5\pi_1 - 0.5\pi_2$ and define compatibility with RHS=0
- in general $\phi' = \phi - 1/n \sum_i \pi_i$
- sum of weights is 0 and **only projections can have negative weight** (otherwise 1st item false)

weighting

- **Def:** n-ary weighting ϕ is
  a formal linear combination of operations

$$\phi = \sum_{f:D^n \to D} \phi(f) f, \quad \text{where} \sum_f \phi(f) = 0 \text{ and } \phi(f) < 0 \Rightarrow f = \pi_i$$

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- wPol , wInv defined analogously to Pol, Inv.

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- Weighting can be superposed with operations in a natural way
- **Example:**
  - binary $\phi = 0.3\max + 0.2\pi_1 - 0.5\pi_2$
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$$\begin{aligned}
\phi(\pi_3, \max{}_{123}) &= 0.3 \max(\pi_3, \max{}_{123}) + 0.2\pi_1(\pi_3, \max{}_{123}) \\
&\quad - 0.5\pi_2(\pi_3, \max{}_{123}) \\
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- **Oops,** this is not a weighting
  (negative weight on a non-projection)
  $\rightarrow$ this superposition is improper

one more caveat

- the following two weighted relational clones over $D = \{0, 1\}$ have no nonzero compatible weightings
  - all weighted relations ρ
  - all weighted relations ρ with Feas(ρ) in the smallest relational clone
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- these weighted clones are different
- **Solution:** Define weighting and weighted clone over a fixed (normal) clone
- (and adjust the definition of wInv accordingly)

weighted clones

- **Def:** Let **A** be a clone. A weighted clone over **A** is a set of weightings **W**, whose supports are contained in **A**, and which is closed under
  (1) nonnegative scaling,
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- **Corollary:** if **W** is a set of weightings over **A**, then

$$\mathrm{wClo}^k(\mathbf{W}) = \left\{ \sum_i a_i \phi_i(f_{i1}, \dots, f_{ik_i}) : a_i \geq 0, \phi_i \in \mathbf{W}, f\,ij \in \mathbf{A} \right\}$$
$$\cap \{\text{all } k\text{-ary weightings}\}$$

algebraic part of the wPol–wInv correspondence

Theorem (Cohen, Cooper, Creed, Jeavons, Živný)

If $\mathbf{W}$ is a finitely generated weighted clone over $\mathbf{A}$ then
$\mathbf{W} = \mathrm{wPol}(\mathrm{wInv}_{\mathbf{A}}(\mathbf{W}))$

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Is there a solution with $x, y, z \geq 0$?

$$4x - 5y - 4z = 2$$
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Farkas' lemma: if $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{o}$ unsolvable then
$\exists \mathbf{y}$ such that $\mathbf{y}^T A \leq \mathbf{o}$, $\mathbf{y}^T \mathbf{b} > 0$

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assume $\tau \notin \mathbf{W} = \mathrm{wClo}(\phi_1, \ldots, \phi_k)$ (say n-ary).

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remarks

- many lattices of weighted clones
  - lattice of weighted clones over a fixed clone **A**
  - lattice of weighted clones (neglect **A**)

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  - lattice of weighted clones over a fixed clone **A**
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- for non-finitely generated weighted clones (still on finite domain)
  - we need $\mathbb{R}$ instead of $\mathbb{Q}$
  - we need to consider closed weighted clones

  Fulla, Živný

results and questions

- minimal and maximal clones
- Boolean domain
- nicer weightings
- positive part

weighted Rosenberg

- **Thm:** Creed, Živný; Thapper, Živný
  Every non-trivial weighted clone **W** contains
  a weighting whose support is
  - a set of unary operations (not projection), or
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  - a set of majority operations, or
  - a set of minority operations, or
  - a set of majority operations with total weight 2 and a set of minority operations with total weight 1, or
  - a set of k-ary semiprojections (for some $k \geq 3$)
- 9 minimal weighted clones on $|D| = 2$
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- possibly easier:
  1. find all "fake" weighted clones
     (negative weights on non-projections allowed)
     (btw. **Question:** is there a relational counterpart?)
  2. look at proper weightings in these "weighted clones"

prettier weightings

- **Theorem:** Thapper, Živný; Kolmogorov
  $\forall k \geq 2$ $\forall \mathbf{W}$ over the full clone
  if $\exists \phi \in \mathbf{W}$ whose supp. contains a k-ary symmetric op
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- **Problem:** if **W** is finitely related, is Pos(**W**) necessarily finitely related?

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