Rectangularity Theorem for Conservative Algebras

Libor Barto

2nd table
Office No. 3941
Department of Algebra
Section of Mathematics
Faculty of Mathematics and Physics
Charles University
Prague 8
Prague
Bohemia
Czech Republic
Central Europe
Europe
Eurasia
Earth
Thanks Pierre!

AAA Bedlewo 2010

CSP for algebras

Definition ($\mathrm{CSP}(\mathbf{A})$)

Let $\mathbf{A}$ be a finite algebra.
$\mathrm{CSP}(\mathbf{A})$ is the following decision problem:

INPUT: Formula of the form

$$(x_1, x_2) \in R_1 \;\;\&\;\; (x_3, x_1, x_3, x_4) \in R_2 \;\;\&\;\; x_7 \in R_3 \;\;\& \dots$$

where each R_i is a subpower of $\mathbf{A}$ ($R_1 \leq \mathbf{A}^2$, $R_2 \leq \mathbf{A}^4$, $R_3 \leq \mathbf{A}$)

QUESTION: Is the formula satisfiable?

CSP for algebras

Definition ($\mathrm{CSP}(\mathbf{A})$)

Let $\mathbf{A}$ be a finite algebra.
$\mathrm{CSP}(\mathbf{A})$ is the following decision problem:

INPUT: Formula of the form

$$(x_1, x_2) \in R_1 \;\; \& \;\; (x_3, x_1, x_3, x_4) \in R_2 \;\; \& \;\; x_7 \in R_3 \;\; \& \dots$$

where each R_i is a subpower of $\mathbf{A}$ ($R_1 \leq \mathbf{A}^2$, $R_2 \leq \mathbf{A}^4$, $R_3 \leq \mathbf{A}$)

QUESTION: Is the formula satisfiable?

Theorem (Bulatov, Jeavons, Krokhin '00)

If $\mathbf{A}$ is not a Taylor algebra, then $\mathrm{CSP}(\mathbf{A})$ is NP-complete.

Conjecture ((A stronger) algebraic dichotomy conjecture)

If $\mathbf{A}$ is a Taylor algebra, then $\mathrm{CSP}(\mathbf{A})$ is tractable.

A (big) theorem of Bulatov

Theorem (BJK'00)

To prove the conjecture, we can WLOG assume that $\mathbf{A}$ *is idempotent.*

$\mathbf{A}$ is idempotent

$\Leftrightarrow f(a, a, \ldots, a) = a \quad$ (for all f, a)

$\Leftrightarrow \{a\} \leq \mathbf{A} \quad$ (for all $a \in A$)

A (big) theorem of Bulatov

Theorem (BJK'00)

To prove the conjecture, we can WLOG assume that $\mathbf{A}$ *is idempotent.*

$\mathbf{A}$ is idempotent

$\Leftrightarrow f(a, a, \ldots, a) = a \quad$ (for all f, a)

$\Leftrightarrow \{a\} \leq \mathbf{A} \quad$ (for all $a \in A$)

Theorem (Bulatov'05)

If $\mathbf{A}$ *is a (finite) conservative Taylor algebra, then* $\mathrm{CSP}(\mathbf{A})$ *is tractable.*

$\mathbf{A}$ is conservative

$\Leftrightarrow f(a_1, a_2, \ldots, a_n) \in \{a_1, a_2, \ldots, a_n\} \quad$ (for all $\ldots$)

$\Leftrightarrow B \leq \mathbf{A} \quad$ (for all $B \subseteq \mathbf{A}$)

A new proof

Bulatov's proof:

- Many cases depending on local structure of the algebra
- Difficult, long (80 pages)

Our new proof:

- Uses theory developed with Marcin Kozik (absorption, Prague strategies)
- + Rectangularity theorem
- Natural, short

Taylor algebras

Theorem

Let **A** *be a finite idempotent algebra. TFAE*

(4) **A** *satisfies some nontrivial idempotent Maltsev condition*

(3) HSP(**A**) *contains a two element algebra, whose every operation is a projection*

(2) **A** *is not at the bottom of the interpretability lattice*

Taylor algebras

Theorem

Let **A** *be a finite idempotent algebra. TFAE*

(4) **A** *satisfies some nontrivial idempotent Maltsev condition*

(3) HSP(**A**) *contains a two element algebra, whose every operation is a projection*

(2) **A** *is not at the bottom of the interpretability lattice*

(7) *Taylor 73* **A** *has a Taylor term*

(5) *Hobby, McKenzie 85* HSP(**A**) *omits type* **1**

(6) *Maróti, McKenzie 06* **A** *has a WNU*

(1) *Barto, Kozik 09* **A** *has a cyclic term*

(8) *Siggers 09* **A** *has a Siggers term*

Taylor algebras

Theorem

Let **A** *be a finite idempotent algebra. TFAE*

(4) **A** *satisfies some nontrivial idempotent Maltsev condition*

(3) HSP(**A**) *contains a two element algebra, whose every operation is a projection*

(2) **A** *is not at the bottom of the interpretability lattice*

(7) *Taylor 73* **A** *has a Taylor term*

(5) *Hobby, McKenzie 85* HSP(**A**) *omits type* **1**

(6) *Maróti, McKenzie 06* **A** *has a WNU*

(1) *Barto, Kozik 09* **A** *has a cyclic term*

(8) *Siggers 09* **A** *has a Siggers term*

Definition

Such algebras are called Taylor.

Absorption

Definition

We say that B is an absorbing subuniverse of $\mathbf{A}$ ($\mathbf{A}$ is idempotent), $B \lhd \mathbf{A}$, if $B \leq \mathbf{A}$ and $\exists t \in \mathrm{Clo}(\mathbf{A})$

$$t(A, B, B, \dots, B) \subseteq B, \;\; t(B, A, B, B, \dots, B) \subseteq B, \dots$$

B is a minimal absorbing subuniverse (MAS) of $\mathbf{A}$, $B \lhd\!\lhd \mathbf{A}$, if it is a minimal absorbing subuniverse :)

Absorption

Definition

We say that B is an absorbing subuniverse of $\mathbf{A}$ ($\mathbf{A}$ is idempotent), $B \triangleleft \mathbf{A}$, if $B \leq \mathbf{A}$ and $\exists t \in \mathrm{Clo}(\mathbf{A})$

$$t(A, B, B, \ldots, B) \subseteq B,\ \ t(B, A, B, B, \ldots, B) \subseteq B, \ldots$$

B is a minimal absorbing subuniverse (MAS) of $\mathbf{A}$, $B \triangleleft\!\!\triangleleft \mathbf{A}$, if it is a minimal absorbing subuniverse :)

Theorem (Barto, Kozik'08)

Let $\mathbf{A}, \mathbf{B}$ be Taylor algebras (in the same idempotent variety), let $R \leq \mathbf{A} \times \mathbf{B}$ be subdirect and linked. Then $R = A \times B$, or $\mathbf{A}$ has a proper absorbing subuniverse, or $\mathbf{B}$ has ...

Rectangularity Theorem

Theorem

Let $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ *be fin. conservative Taylor algs from a variety*
$B_i \lhd\!\lhd \mathbf{A}_i$, $R \leq \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ *subdirect*

Assume $R \cap (B_1 \times \cdots \times B_n) \neq \emptyset$

Rectangularity Theorem

Theorem

Let $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ be fin. conservative Taylor algs from a variety
$B_i \lhd\!\lhd \mathbf{A}_i$, $R \leq \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ subdirect
Assume $R \cap (B_1 \times \cdots \times B_n) \neq \emptyset$

Define $i \sim j$ iff $\forall (a_1, \ldots, a_n) \in R \;\; a_i \in B_i \Leftrightarrow a_j \in B_j$
Let $D_1, \ldots, D_k$ be $\sim$-blocks

Rectangularity Theorem

Theorem

Let $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ be fin. conservative Taylor algs from a variety
$B_i \lhd\lhd \mathbf{A}_i$, $R \le \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ subdirect
Assume $R \cap (B_1 \times \cdots \times B_n) \neq \emptyset$

Define $i \sim j$ iff $\forall (a_1, \dots, a_n) \in R \;\; a_i \in B_i \Leftrightarrow a_j \in B_j$
Let $D_1, \dots, D_k$ be $\sim$-blocks

Then

a tuple $\mathbf{b} = (b_1, \dots, b_n) \in B_1 \times \cdots \times B_n$ is in R
iff
the restriction of $\mathbf{b}$ to D_j is in the projection of R to D_j
(for all $j = 1, \dots, k$).