Alg-universality of set functors

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We say that the category of graphs (resp. distributive lattices, topological spaces) is group-universal.

Generalization group → monoid

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Generalization monoid $\rightarrow$ category

Definition

Let $\mathbf{L}, \mathbf{K}$ be categories. A functor $\Phi : \mathbf{L} \rightarrow \mathbf{K}$ is *full embedding*, if it is bijective on hom-sets, i.e. for every pair A, B of $\mathbf{L}$-objects, the mapping $\Phi : \mathrm{Hom}_{\mathbf{L}}(A, B) \rightarrow \mathrm{Hom}_{\mathbf{K}}(\Phi A, \Phi B)$ is a bijection.

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Example

Monoid = one object category.
$\mathbf{K}$ is monoid-universal iff $\forall \mathbf{L}$ one object category $\exists \Phi : \mathbf{L} \to \mathbf{K}$ full embedding.

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- universal, if $\forall \mathbf{L}$ cocretizable category, $\exists \Phi : \mathbf{L} \to \mathbf{K}$ full embedding

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- universal, if $\forall \mathbf{L}$ cocretizable category, $\exists \Phi : \mathbf{L} \to \mathbf{K}$ full embedding
- hyper-universal, if $\forall \mathbf{L}$ category $\exists \Phi : \mathbf{L} \to \mathbf{K}$ full embedding

Remarks

- Every small category can be fully embedded into some $\mathrm{Alg}(\Sigma)$. Hence alg-universality is a stronger property than monoid-universality. But, no natural example (a variety, a quasivariety) of monoid-universal category, which is not alg-universal, is known.

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- The statement "alg-universality = universality"' is equivalent to "the class of measurable cardinals is a set". (Hedrlín, Kučera, Pultr (73))
- No concrete category is hyper-universal. Every concrete universal category has a factor which is hyper-universal (follows from Trnková (66) and Kučera (71)). We haven't described the factor for any universal category.

Examples

- Group-universal categories: Extensive survey: Fung, Kegel, Strambach, *Gruppenuversalität und homogenisierbarkeits*. Ann. Math. Pur. Appl. 141, 1985.

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- Group-universal categories in a stronger sense:
  - Clones Barkhudaryan, Trnková (02)
  - Set endofunctors Barto, Zima (05)

Examples

- Alg-universal categories:
  - $\mathrm{Alg}(\Sigma)$, where $\sum \Sigma \geq 2$; (undirected) graphs Hedrlín, Pultr (66), Vopěnka
  - Semigroups Hedrlín, Lambek (69), Koubek, Sichler (84)
  - $(0,1)$-lattices Grätzer, Sichler (70), Goralčík, Koubek, Sichler (90)
  - Integeral domains Fried, Sichler (77)

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- Universal categories:
  - Hypergraphs Hedrlín, Kučera (80)
  - Topological spaces and open continuous maps Pultr, Trnková (80)
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Pultr, Trnková, *Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories*, 1980.

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Set functor = endofunctor of the category **Set** of all sets and mappings

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The category of all set set functors is not legitimate (too many objects).
Natural legitimate subcategories

- The category of κ-accesible set functors (example: free functors of varieties, every operation of arity less than κ)
- The category of accessible set functors

Set functors vs. Clones

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A variety can be described in terms of a finitary monad over **Set**.
Finitary monad = triple (F, μ, ν), where F is finitary set functor, $\nu : Id \to F$, $\mu : F^2 \to F$ + axioms.
Monad homomorphisms = natural transformations which preserve μ, ν

Monad homomorphisms correspond precisely to interpretations.

Main Theorem and Problems

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Problems:

- Are accessible set functors universal?
- Are set functors hyper-universal?
- Are clones alg-universal?

Thank you for your attention!