Height one identities

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Is it the end of UA?

UA (universal algebra) and CSP (constraint satisfaction problems)

- connection discovered about 20 years ago
- central topic in UA
- UA in top TCS conferences (FOCS, STOC) and journals (JACM, SICOMP)
- the main problem in CSP solved [Bulatov'07]; [Zhuk'07]
- **Is it the end of the great period for UA?**

Particularly promising: PCSP (Promise CSP)

- active both in TCS (long time) and UA (last 2 years)
- UA relevant
- UA can definitely contribute
- **this talk:** methods from other fields in UA

Height one identities, CSP, PCSP

- (identification) minor of $f : A^n \to A$ is
  an operation $g : A^m \to A$ defined by

$$g(x_1, \ldots, x_m) = f(\text{ variables })$$

- height one identity is of the form

$$f(\text{ variables }) = g(\text{ variables })$$

- i.e. equality between identification minors of f and g
- **Note:** operation symbols on both sides
  e.g. $f(x, x, y) = x$ is not height one
- **Note:** makes sense for $f, g : A^n \to B$

- for finite relational structure $\mathbb{A}$
  - $\mathrm{CSP}(\mathbb{A})$: given $\mathbb{X}$ find $\mathbb{X} \to \mathbb{A}$
  - ... a computational problem, one for each $\mathbb{A}$
  - **Example:** Find a 3-coloring of a graph (for $\mathbb{A} = \mathbb{K}_3$)
  - $\mathrm{Pol}(\mathbb{A}) = \{f : \mathbb{A}^n \to \mathbb{A}\}$ polymorphisms
  - **Fact:** it is a clone
- complexity of $\mathrm{CSP}(\mathbb{A})$ depends only on
  - $\mathrm{Pol}(\mathbb{A})$ [Jeavons'98]
  - identities in $\mathrm{Pol}(\mathbb{A})$ [Bulatov, Jeavons, Krokhin'05]
  - height one identities in $\mathrm{Pol}(\mathbb{A})$ [B, Opršal, Pinsker'17]
- $\mathrm{CSP}(\mathbb{A})$ is
  - **hard** if polymorphisms don't satisfy some "nontrivial" height one identities
  - **easy** if they do
  - here "nontrivial" means not satisfiable by projections [Bulatov'17]; [Zhuk'17]

- for finite relational structures $\mathbb{A}, \mathbb{B}$ with $\mathbb{A} \to \mathbb{B}$
  - $\mathrm{PCSP}(\mathbb{A})$: given $\mathbb{X}$ such that $\mathbb{X} \to \mathbb{A}$ find $\mathbb{X} \to \mathbb{B}$
  - ... a computational problem, one for each pair $\mathbb{A}, \mathbb{B}$
  - **Example:** Find a 4-coloring of a 3-colorable graph
  - $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) = \{f : \mathbb{A}^n \to \mathbb{B}\}$ polymorphisms
  - **Observe:** general composition does not make sense
  - **Fact:** closed under identification minors
    (it is a clonoid(?), minion(?), ...)
- complexity of $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ depends only on
  - $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ [Brakensiek, Guruswami'16]
  - height one identities in $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ [Bulín, Opršal]
- $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is
  - **hard** if polymorphisms don't satisfy
    some "nontrivial" height one identities
  - **easy** if they do
  - here "nontrivial" means ???

Cyclic monotone Boolean operations

probabilistic method, analysis of Boolean functions

Boolean operation $f : \{0,1\}^n \to \{0,1\}$ is

- cyclic if $f(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$
- fully symmetric if $f(x_1, x_2, \ldots, x_n) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ for each $\pi \in S_n$
- threshold if it equals thr_α for some α where

$$\mathrm{thr}_\alpha(x_1, \ldots, x_n) = 1 \text{ iff } \sum x_i > \alpha n$$

- monotone if it preserves $\leq$ where $0 \leq 1$

Note: threshold $=$ monotone $+$ fully symmetric

Theorem ([B])

For each k there exists l such that every cyclic monotone Boolean operation of arity $n \geq l$ has an identification minor of arity $\geq k$ which is a threshold operation.

- ∞-many threshold polymorphisms $\Rightarrow$ tractability of PCSP [Brakensiek,Guruswami'16]
- theorem reduces the gap between hardness and tractability for monotone Boolean PCSPs
- height one identities of "permutation type" seems important
- cyclic operations: especially simple + useful in CSP and vCSP

Let $f : \{0,1\}^n \to \{0,1\}$ and $p \in [0,1]$

- choose $x_1, \ldots, x_n \in \{0,1\}$ independently
  - $x_i = 1$ with probability p
  - $x_i = 0$ with probability $1 - p$
- $E_f(p) =$ expected value of $f(x_1, \ldots, x_n)$
- $I_f(p, i)$ influence of the i-th variable
  $=$ probability that $f(x_1, \ldots, x_n)$ changes when x_i is changed
- $I_f(p) := \sum_i I_f(p, i)$ total influence

Theorem ("Russo's Lemma")

$E_f'(p) = I_f(p)$

Theorem ("KKL Theorem" [Kahn, Kalai, Linial'88])

$\exists i \quad I_f(p, i) \geq C \; E_f(p)(1 - E_f(p)) \, \frac{\log n}{n}$

Proof 1/2

Proving: Cyclic monotone $f : \{0,1\}^n \to \{0,1\}$ of sufficiently large arity n has a threshold minor of arity ≥ 10.

Russo's Lemma: $E_f'(p) = I_f(p)$
KKL Theorem: $\exists i \;\; I_f(p,i) \geq C \; E_f(p)(1 - E_f(p)) \; \log n/n$

- take p such that $E_f(p) = 0.5$, say $E_f(0.36) = 0.5$
- f cyclic so $I_f(p,i) = I_f(p,j)$ so $I_f(p) = nI_f(p,i)$
- Russo+KKL: $E_f'(p) = I_f(p) \geq CE_f(p)(1 - E_f(p)) \; \log(n)$
- if $0.00001 \leq E_f(p) \leq 0.99999$ then $E_f'(p) \geq D \log(n)$
- n large $\Rightarrow$
  - if $p < 0.35$ then $E_f(p) < 0.00001$
  - if $p > 0.37$ then $E_f(p) > 0.99999$

$p < 0.35 \Rightarrow E_f(p) < 0.00001 \qquad p > 0.37 \Rightarrow E_f(p) > 0.99999$

- choose a random 10-ary minor of f
  ie. define $g(x_1, \ldots, x_{10}) = f(y_1, \ldots, y_n)$ where
  y_i are chosen uniformly independently from $\{x_1, \ldots, x_{10}\}$
- **Aim**: $\mathrm{P}(g = \mathrm{thr}_{0.35}) > 0$
- $\mathrm{Exp}(g(1,1,1,0,0,0,\ldots,0)) = E_f(3/10) < 0.00001$
- $\mathrm{Exp}(g(1,1,1,1,0,0,\ldots,0)) = E_f(4/10) > 0.99999$
- Expected value of

  $$V := g(1,1,1,0,0,\ldots,0) + g(1,1,0,1,0,\ldots,0) + \cdots + (1 - g(1,1,1,1,0,\ldots,0)) + (1 - g(1,1,1,0,1,0,\ldots,0)) + \ldots$$

  is at most $\binom{10}{3}0.00001 + \binom{10}{4}0.00001 < 1$
- So $\mathrm{P}(V = 0) > 0$
- But $\mathrm{P}(V = 0) = \mathrm{P}(g = \mathrm{thr}_{0.35})$

Blockers

Topological combinatorics, PCP theory

Let $f : [3]^n \to [5]$ where $[i] = \{1, 2, \ldots, i\}$

- $f \in \mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$ if
  $f(x_1, \ldots, x_n) \neq f(y_1, \ldots, y_n)$ whenever $(\forall i)\ x_i \neq y_i$
- subset of coordinates $I \subseteq \{1, \ldots, n\}$ blocks $h : [3]^2 \to [5]$
  if no minor of the form

$$g(x, y) = f(z_1, \ldots, z_n) \text{ with } z_i = x \text{ for } i \in I$$
$$\text{and } z_i \in \{x, y\} \text{ otherwise}$$

  is equal to h

Theorem ([Dinur, Regev, Smyth'05] + [B] + [Opršal])

Each $f \in \mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$ has a "small" subset of coordinates I that blocks some $h : [3]^2 \to [5]$. (small means e.g. $|I| \leq 10^6$)

- "unique blocking with singleton I" **characterizes** NP-hardness of CSP:
  $\mathrm{CSP}(\mathbb{A})$ is NP-hard iff there exists a set of binary functions $\exists H$ such that for each $f \in \mathrm{Pol}(\mathbb{A})$ there exists a unique i such that $\{i\}$ blocks each $h \in H$.
- blocking with larger I (as in Theorem) + some form of uniqueness **sufficient** for NP-hardness of PCSP
- Theorem is a substantial part of the proof that it is NP-hard to 5–color a 3–colorable graph

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Each $f \in \mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$ has a "small" subset of coordinates I that blocks some $h : [3]^2 \to [5]$. (small means e.g. $|I| \leq 10^6$)

- topological combinatorics founded by a proof of Kneser's conjecture [Lovász'78]
- many alternative proofs of Kneser's conjecture
  [Bárány'78], [Greene'02], [Matoušek'04], ...
- Theorem + PCP theory → NP-hardness of PCSP(NAE,k-NAE) [Dinur, Regev, Smyth'05]
- universal algebraic version [B]
- $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$ is NP-hard [Opršal]

- k–sphere $S^k = \{\mathbf{x} \in \mathbb{R}^{k+1} : \|\mathbf{x}\| = 1\}$
- open hemisphere centered at $\mathbf{a}$ = $H(\mathbf{a}) = \{\mathbf{x} \in S^k : \mathbf{a} \cdot \mathbf{x} > 0\}$
- great $(k-1)$–sphere in S^k = $\{\mathbf{x} \in S^k : \mathbf{a} \cdot \mathbf{x} = 0\}$

Theorem (LSB theorem [Lusternik, Schnirelmann'30])

If S^k is covered by $k+1$ open sets, then one of these sets contains both $\mathbf{a}$ and $-\mathbf{a}$ for some $\mathbf{a}$.

$f : A^6 \to B$ is Olšák operation if

$$
\begin{aligned}
t(y, x, x,\ x, y, y) &= \\
t(x, y, x,\ y, x, y) &= \\
t(x, x, y,\ y, y, x)&
\end{aligned}
$$

Theorem ([Opršal])

There is no Olšák operation in $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$

Proof: Otherwise

$$
\begin{gathered}
t(1, 2, 3,\ 2, 3, 1), t(2, 3, 1,\ 3, 1, 2), t(3, 1, 2,\ 1, 2, 3), \\
t(2, 1, 1,\ 1, 2, 2), t(3, 2, 2,\ 2, 3, 3), t(1, 3, 3,\ 3, 1, 1)
\end{gathered}
$$

would form a 6-clique in $\mathbb{K}_5$

Theorem: Each $f \in \mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$ has a small subset of coordinates I that blocks some $h : [3]^2 \to [5]$.

- take $f : [3]^n \to [5] \in \mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$
- $k := \#$ of binary operations in $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$ minus 1
- distribute n points $\mathbf{p}_1, \ldots, \mathbf{p}_n$ on S^k in general position, ie. no $k+1$ points lie on s great $(k-1)$-sphere
- for $Q \subseteq [n]$ let $f[Q]$ be the binary minor $f(x/y, \ldots)$ where x's are at positions in Q and y's are at the other positions
- for each binary $h \in \mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$ let
  $U_h = \{\mathbf{a} \in S^k : f[\{i : \mathbf{p}_i \in H(\mathbf{a})\}] = h\}$
- LSB theorem: some U_h contains $\mathbf{a}$ and $-\mathbf{a}$ for some $\mathbf{a}$
  cheating!

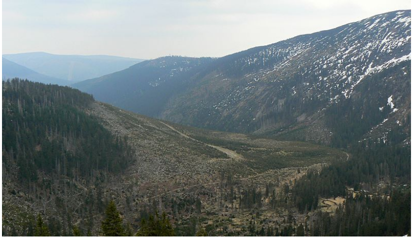
- let's ignore it (can be repaired)
- we have $\mathbf{a} \in S^k$ such that $f[\{i : \mathbf{p}_i \in H(\mathbf{a})\}] = h = f[\{i : \mathbf{p}_i \in H(-\mathbf{a})\}]$
- after reordering of variables

$$\begin{aligned} f(y, y, \dots, y, \quad & x, x, \dots, x, \quad & y, y, \dots, y) = h \\ f(y, y, \dots, y, \quad & y, y, \dots, y, \quad & x, x, \dots, x) = h \end{aligned}$$

  where the initial segment of x's is small
  since $\mathbf{p}_i$'s are in general position
- this set of coordinates blocks h since otherwise

$$f(x, x, \dots, x, \quad x/y, \dots, x/y, \quad x/y, \dots, x/y) = h$$

  and a suitable 6-ary minor would be an Olšák operation



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Register and pay by June 15th

Summary

- universal algebra can help in a large part of mathematics
- there is so much beautiful math useful in universal algebra

Reading

- G. Kalai: Boolean Functions: Influence, threshold and noise
- R. O'Donnell: Analysis of Boolean functions
- M. de Longueville: 25 years proof of the Kneser conjecture - The advent of topological combinatorics
- J. Matoušek: Using the Borsuk-Ulam Theorem

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Thank you!