Reconstructing subproducts from projections

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Recall the near unanimity (NU) identities

$$f(y,x,x,\dots,x) \approx f(x,y,x,\dots,x) \approx \cdots \approx f(x,x,\dots,x,y) \approx x$$

$\mathrm{NU}(l)$: near unanimity term of arity $l \geq 3$

- [Baker-Pixley] A variety has $\mathrm{NU}(k+1)$
  iff subproducts are determined by k-fold projections
- [Bergman] **If** a variety has $\mathrm{NU}(k+1)$,
  then consistent systems of k-ary relations
  are k-fold projections of subproducts
- [Our result] A variety has $\mathrm{NU}(k+2)$
  iff consistent systems of k-ary relations
  are k-fold projections of subproducts

This talk: $k = 2$

[K. Baker, A. Pixley'75: Polynomial interpolation and the Chinese remainder theorem for algebraic systems]

Theorem

Let $\mathcal{V}$ be a variety. TFAE.

(i) *$\mathcal{V}$ has $\mathrm{NU}(3)$.*

(ii) *Every $R \leq \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ with $\mathbf{A}_i \in \mathcal{V}$ is uniquely determined by the system $(\mathrm{proj}_{ij}(R))_{i,j\in[n],i\neq j}$*

Item (ii) rephrased

- for every $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathcal{V}$
- for every $(P_{ij})_{i,j\in[n],i\neq j}$ where $P_{ij} \leq \mathbf{A}_i \times \mathbf{A}_j$
- there exists at most one $R \leq \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ such that $(\forall i,j)\ P_{ij} = \mathrm{proj}_{ij}(R)$

What about at least?

Binary system over $\mathcal{V}$

- $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathcal{V}$
- $(P_{ij})_{i,j \in [n]}$ where $P_{ij} \leq \mathbf{A}_i \times \mathbf{A}_j$ (always $i \neq j$)

Witnessing relation: $R \leq \mathbf{A}_1 \times \ldots \mathbf{A}_n$ with $(\forall i, j)\ P_{ij} = \mathrm{proj}_{ij}(R)$

Baker-Pixley: A variety $\mathcal{V}$ has NU(3) iff
every binary system over $\mathcal{V}$ has at most one witnessing relation

Sometimes: clearly no witnessing relation exists, e.g.:

- $P_{12} = \{(1,1)\}$, $P_{21} = \{(1,2)\}$
- $P_{12} = P_{23} = \{(1,1),(2,2)\}$, $P_{13} = \{(1,2),(2,1)\}$

Definition

(P_{ij}) is consistent if

- $(\forall i, j)\ P_{ij} = P_{ji}^{-1}$
- $(\forall i, j, k)\ (\forall a_i a_j \in P_{ij})\ (\exists a_k)\ \ a_i a_k \in P_{ik}$ and $a_j a_k \in P_{jk}$

[G. Bergman'77: On the existence of subalgebras of direct products with prescribed d-fold projections]

Theorem

Let $\mathcal{V}$ be a variety. Then (i) implies (ii).

(i) *$\mathcal{V}$ has $\mathrm{NU}(3)$.*

(ii) *Every consistent binary system (P_{ij}) over $\mathcal{V}$ has a witnessing relation.*

Remarks:

- Bergman gave strengthening (ii') of (ii) and proved (i) $\Leftrightarrow$ (ii')
- Very similar result later obtained in the context of CSPs

  [Feder, Vardi'98], [Jeavons, Cohen, Cooper'98]

Our result

[Barto, Kozik, Tan, Valeriote: Sensitive instances of CSPs, submitted]

Theorem

Let $\mathcal{V}$ be a variety. TFAE.

(i) *$\mathcal{V}$ has $\mathrm{NU}(4)$.*

(ii) *Every consistent binary system (P_{ij}) over $\mathcal{V}$ has a witnessing relation.*

For a local version (concerning a single algebra):

Definition

An algebra $\mathbf{A}$ has local $\mathrm{NU}(l)$ if for every finite $F \subseteq A$ there exists an l-ary term operation t_F of $\mathbf{A}$ such that $t_F(b, a, \dots, a) = t_F(a, b, a, \dots, a) = \cdots = t_F(a, \dots, a, b) = a$ for every $a, b \in F$.

Theorem (Local Baker-Pixley)

Let $\mathbf{A}$ *be an idempotent algebra. TFAE.*

(i) $\mathbf{A}$ *has local NU(3).*

(ii) *Every binary system over* $\{\mathbf{A}\}$ *has at most one witnessing relation.*

Theorem (Local version of our result)

Let $\mathbf{A}$ *be an idempotent algebra. TFAE.*

(i) $\mathbf{A}$ *has local NU(4).*

(ii) *Every binary system over* $\{\mathbf{A}^2\}$ *has at least one witnessing relation.*

Remark: Idempotency necessary, square in $\mathbf{A}^2$ as well.

Proof

Theorem (Local version of our result)

Let $\mathbf{A}$ *be an idempotent algebra. TFAE.*

(i) $\mathbf{A}$ *has local NU(4).*

(ii) *Every binary system over* $\{\mathbf{A}^2\}$ *has at least one witnessing relation.*

(ii) $\Rightarrow$ (i)

- careful choices of systems give "very local" $\mathrm{NU}(4)$'s
- local $\mathrm{NU}(4)$'s can be assembled from these [Horowitz'13]

(i) $\Rightarrow$ (ii)

- candidate witness (the largest if any exists):
  $R = \{a_1 a_2 \dots a_n : (\forall i, j)\ a_i a_j \in P_{ij}\}$
- enough to show: $(\forall i, j)(\forall a_i a_j \in P_{ij})$ there is an extension in R

Proof

Theorem (Local version of our result)

Let **A** *be an idempotent algebra. TFAE.*

(i) **A** *has local NU(4).*

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(i) $\Rightarrow$ (ii)

- predecessor: a version for finite algebras [BK]
- main tool for the predecessor: a loop lemma
- main tool for this result: an infinite loop lemma

Here: $S \subseteq T \leq \mathbf{B}^2$, S "locally absorbs" T, $\mathbf{B}$ idempotent

[$\sim$ Olšák'17] If S is symmetric and $\Delta_A \subseteq T$, then $S \cap \Delta_A \neq \emptyset$.
[BKTV] If S has a directed cycle and $\Delta_A \subseteq T$, then $S \cap \Delta_A \neq \emptyset$.
[BKTV] if S has a long d.walk and $\Delta_A \cup S^{-1} \subseteq T$, then $S \cap \Delta_A \neq \emptyset$.

Fix $a_1, a_2, a_3 \in B$ and **assume**
$\exists a_4$ such that $a_i a_j \in P_{ij}$ for $(i,j) \in \mathcal{I}$ ($a_1 a_2 a_3 a_4$ works for $\mathcal{I}$)
$\exists a_4$ such that $a_i a_j \in P_{ij}$ for $(i,j) \in \mathcal{J}$
Consider
$S = \{a_4 a_4' : a_1 a_2 a_3 a_4$ works for $\mathcal{I}$, $a_1 a_2 a_3 a_4'$ works for $\mathcal{J}$ $\}$
$T = \{b_4 b_4' : (\exists b_1, b_2, b_3)\ b_1 b_2 b_3 b_4$ work for $\mathcal{I}$, $\dots\}$
Then
S locally absorbs T (because of local NU)
if S and T satisfies ... we get $a_4 a_4 \in S$ for some a_4
ie. $a_1 a_2 a_3 a_4$ works for $\mathcal{I} \cup \mathcal{J}$

Open problem

Of interest for ∞-domain CSPs:

Question

Assume $\mathbf{A}$ *is oligomorphic core and* $\mathbf{A}$ *has a quasi-*$\mathrm{NU}(4)$*, i.e.,*

$t(y,x,x,x) \approx t(x,y,x,x) \approx t(x,x,y,x) \approx t(x,x,x,y) \approx t(x,x,x,x)$

Does every binary system over $\{\mathbf{A}\}$ *necessarily have at least one witnessing relation?*

Remarks:

- quasi-$\mathrm{NU}(4) \Rightarrow$ local $\mathrm{NU}(4)$, but not idempotent
- the loop lemma with quasi-absorption does not work

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Thank you!