# A welcome conservative talk

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# CSP over algebra

**A** ... fixed finite idempotent algebra

**Definition ($\mathrm{CSP}(\mathbf{A})$ - Constraint Satisfaction Problem over A)**

INSTANCE:

$V$ ... set of variables

$\mathcal{C}$ ... set of constraints

**constraint** ... pair $(\mathbf{x}, R)$, where

$\mathbf{x} = (x_1, \ldots, x_k) \in V^k$ (**scope**) assume $x_1, \ldots, x_k$ distinct

$R \leq \mathbf{A}^k$ (**constraint relation**)

QUESTION: Does there exist a solution?

**solution** ... mapping $f : V \to A$ such that $f(\mathbf{x}) \in R$ for each $(\mathbf{x}, R) \in \mathcal{C}$

# Conservative CSPs

$\mathbf{A}$ is conservative, if $B \leq \mathbf{A}$ for every $B \subseteq A$

**Theorem (Bulatov)**

*Let $\mathbf{A}$ be conservative. If $\mathbf{A}$ is Taylor, then $\mathrm{CSP}(\mathbf{A})$ is tractable. (And otherwise $\mathrm{CSP}(\mathbf{A})$ is NP-complete. )*

Proof is long, complicated case analysis

2 new proofs

(a) using absorption, Prague strategies and one new algebraic result about conservative algebras B
(today)

(b) using Bulatov's colors and Maróti's retraction trick Bulatov
(tommorow)

# Outline

Algorithm (simplified):

- Make the instance $(2,3)$-minimal (old stuff)
- Find (using walking) a 1-minimal subinstance going through minimal absorbing subuniverses (old stuff)
- Use “Rectangularity theorem” (the only new stuff) to either find a solution, or shrink the instance

Outline:

- Consistency notions ($(k,l)$-minimality)
- Walking and Prague strategies
- Absorption, finding nice MASes
- Rectangularity theorem – baby case
- Algorithm for binary constraints, simplified
- Algorithm for binary constraints, real
- Rectangularity theorem

# Consistency notions

## Definition (1-minimality)

An instance of $\mathrm{CSP}(\mathbf{A})$ is 1-minimal, if

- for every $x \in V$ there is a unique constraint with scope $(x)$ $\ldots ((x), S_x)$
- for every $((x_1, \ldots, x_k), R) \in \mathcal{C}$ and every $i$, the projection of $R$ to the $i$-th coordinate is equal to $S_{x_i}$
  i.e. $R$ is subdirect in $S_{x_1} \times \cdots \times S_{x_k}$

# Consistency notions

## Definition (1-minimality)

An instance of $\mathrm{CSP}(\mathbf{A})$ is 1-minimal, if

- $\forall x \in V$ unique $((x), S_x) \in \mathcal{C}$
- $\forall ((x_1, \ldots, x_k), R) \in \mathcal{C} \ \Rightarrow \ R \leq_S S_{x_1} \times \cdots \times S_{x_k}$

## Definition ((2,3)-minimality, standard definition)

A 1-minimal instance of $\mathrm{CSP}(\mathbf{A})$ is $(2,3)$-minimal, if

- for every $x_1 \neq x_2 \in V$ there is a unique constraint with scope $(x_1, x_2) \ldots ((x_1, x_2), S_{x_1,x_2})$ (let $S_{x,x} = \{(a,a) : a \in S_x\}$)
- for every $(\mathbf{x}, R) \in \mathcal{C}$ whose scope contains $x_1, x_2$,the projection of $R$ to the corresponding coordinates is equal to $S_{x_1,x_2}$
- every triple of variables is within a scope of some constraint

# Consistency notions

## Definition (1-minimality)

An instance of $\mathrm{CSP}(\mathbf{A})$ is 1-minimal, if

- $\forall x \in V$ unique $((x), S_x) \in \mathcal{C}$
- $\forall ((x_1, \dots, x_k), R) \in \mathcal{C} \Rightarrow R \leq_S S_{x_1} \times \dots \times S_{x_k}$

## Definition ((2,3)-minimality, essentially the same)

A 1-minimal instance of $\mathrm{CSP}(\mathbf{A})$ is $(2,3)$-minimal, if

- for every $x_1 \neq x_2 \in V$ there is a unique constraint with scope $(x_1, x_2) \dots ((x_1, x_2), S_{x_1,x_2})$ (let $S_{x,x} = \{(a,a) : a \in S_x\}$)
- for every $(\mathbf{x}, R) \in \mathcal{C}$ whose scope contains $x_1, x_2$,the projection of $R$ to the corresponding coordinates is equal to $S_{x_1,x_2}$
- for every $x_1, x_2, x_3 \in V$ and every $(a, b) \in S_{x_1,x_2}$ there exists $c$ such that $(a, c) \in S_{x_1,x_3}$ and $(b, c) \in S_{x_2,x_3}$

# Walking in 2-minimal instances

If $R \subseteq A^2$ and $B \subseteq A$, let $R^+[B] = \{c \in A : \exists b \in B \ (b, c) \in R\}$

Let $x, y \in V$ and $\emptyset \neq B \leq \mathbf{S}_x$, $\emptyset \neq C \leq \mathbf{S}_y$. We write

$$(x, B) \leq_1 (y, C) \quad \text{iff} \quad S^+_{x,y}[B] = C$$

$\leq$ ... the transitive closure of $\leq_1$ $\Rightarrow$ qoset QOSET

Write $(x, B) \sim (y, C)$, if $(x, B) \leq (y, C) \leq (x, B)$

**Definition (Prague strategy, for 2-minimal instances)**

A 2-minimal instance is a Prague strategy, if
$(x, B) \sim (y, C)$ implies $(x, B) \leq_1 (y, C)$

In general, Prague strategy = 1-minimal instance + ...
Our definition suffices for instances with at most binary constraints

**Fact**

*Every $(2, 3)$-minimal instance is a Prague strategy.*

# Absorption

## Definition

$B$ is an absorbing subuniverse of $\mathbf{T}$, if $\mathbf{T}$ has a term $t$ such that for every coordinate $i$

$$t(B, B, \ldots, B, T, B, B, \ldots, B) \subseteq B \quad (T \text{ is at the } i\text{-th coordinate})$$

Notation: $B \triangleleft \mathbf{T}$.

## Definition

$B$ is a minimal absorbing subuniverse of $\mathbf{T}$ if $B \triangleleft \mathbf{T}$ and $\mathbf{B}$ has no proper absorbing subuniverses. Notation: $B \lll \mathbf{T}$.

## Fact

*If $P$ is a Prague strategy, $B_x \triangleleft \mathbf{S}_x$ and the restriction of $P$ to $B$'s is 1-minimal, then this restriction is a Prague strategy.*

# Subalgorithm

## Fact

*Let $P$ be a Prague strategy over $\mathbf{A}$. There exist $E_x \lll \mathbf{S}_x$, $x \in V$ such that the restriction of $P$ to $E$'s is a Prague strategy. Moreover, there is a P-time algorithm for finding such $E$'s.*

## Proof.

- Consider subqoset $\mathrm{AbsQoset}$ of $\mathrm{QOSET}$ formed by all pairs $(x, B)$ such that $B$ is a proper absorbing subuniverse of $\mathbf{S}_x$ (fact: it is an upset)
- Find a maximal component $\{(x, R_x) : x \in W\}$ of $\mathrm{AbsQoset}$ (where $W \subseteq V$). Define $R_x = S_x$ for $x \in V - W$
- The restriction of $P$ to $R$'s is 1-minimal (because $P$ is a Prague strategy)
- This restriction is a Prague strategy (from the previous fact)
- Restrict and repeat

# Subalgorithm

**Fact**

*Let $P$ be a Prague strategy over $\mathbf{A}$. There exist $E_x \lll \mathbf{S}_x$, $x \in V$ such that the restriction of $P$ to $E$'s is a Prague strategy.*

*(We will only need that the restriction is 1-minimal.)*

*Moreover, there is a P-time algorithm for finding such $E$'s.*

- Works for Prague strategies over arbitrary idempotent algebra (not necessarily conservative, not necessarily Taylor)
- Proves that NU implies width $(2, 3)$

# Rectangularity theorem – baby case

## Fact (Inspiration: Bulatov's original proof)

*Let*

- $\mathbf{T}_1, \mathbf{T}_2$ *be conservative Taylor algebras*
- $R \leq_S \mathbf{T}_1 \times \mathbf{T}_2$,
- $B_1 \lll \mathbf{T}_1,\ B_2 \lll \mathbf{T}_2$
- $R \cap (B_1 \times B_2) \neq \emptyset$
- $\exists a_1 \in T_1 - B_1\ \exists b_2 \in B_2\ \ (a_1, b_2) \in R$

*Then* $B_1 \times B_2 \subseteq R$.

# Proof of the baby case

**Proof.**

- Draw a potato picture. "linked" below means connected on the picture.
- Let $S = R \cap (B_1 \times B_2)$
- $S \leq_S B_1 \times B_2$ (as the projection of $S$ to the first coordinate absorbs $B_1$)
- Let $C =$ all elements not $R$-linked to $a_1$
- If $C = \emptyset$, then
  - $S$ is linked (use the fact that connectivity is absorbed)
  - $S = B_1 \times B_2$ (using Absorption Theorem)
- If $C \neq \emptyset$, then $C \triangleleft \mathbf{B}_1$ (using conservativity)

$\square$

# Algorithm for binary constraints, simplified

Assume that

- We can solve instances over smaller domains in P-time
- All constraints are at most binary

The algorithm:

1. Find an equivalent $(2, 3)$-minimal instance $P$
2. **Assume that every $\mathbf{S}_x$ has a proper absorbing subuniverse**
3. Find $\{E_x : x \in V\}$ such that $E_x \lll \mathbf{S}_x$ and the restriction of $P$ to $E$'s is 1-minimal (use the subalgorithm)
4. Find a partition $V = V_1 \cup \dots V_l$ such that $(x, E_x) \sim (y, E_y)$ whenever $x, y$ are in the same $V_i$ (strands)
5. Using inductive assumption, find partial solutions $f_i : V_i \longrightarrow A$.
   - If some $f_i$ does not exist, then we can delete $E_x$ from $S_x$, $x \in V_i$ and start again
   - If all $f_i$ exist, then $\cup f_i$ is a solution by Rectangularity theorem, baby case

# Algorithm for binary constraints

1. Find an equivalent $(2,3)$-minimal instance $P$
2. Consider the subqoset $\mathrm{NafaQoset}$ of $\mathrm{QOSET}$ formed by $(x, B)$ such that **B** has a proper absorbing subuniverse
3. If $\mathrm{NafaQoset}$ is nonempty
   - Find a maximal component $\{(x, D_x) : x \in W\}$ of $\mathrm{NafaQoset}$
   - Let $Q$ be the restriction of $P$ to $D$'s
   - Find $E$'s for the instance $Q$ as before
   - Solve in strands as before, if impossible, delete $E_x$ and go to 1.
   - Delete $D_x - E_x$, $x \in W$ and go to 1.
4. If $\mathrm{NafaQoset}$ is empty, then we are Mal'tsev (use WNU)

For general constraints:

- The algorithm is the same
- The proof of correctness requires Rectangularity Theorem in full generality:

# Rectangularity theorem

## Theorem

*Let*

- $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n$ *be conservative Taylor algebras*
- $R \leq_S \mathbf{T}_1 \times \mathbf{T}_2 \times \cdots \times \mathbf{T}_n$,
- $B_1 \lll \mathbf{T}_1$, $B_2 \lll \mathbf{T}_2$, $\ldots$, $B_n \lll \mathbf{T}_n$
- $R \cap (B_1 \times B_2 \times \cdots \times B_n) \neq \emptyset$

*Define*

- $i \sim j$ *if* $R|^{+}_{i,j}[B_i] = B_j$ *and* $R|^{+}_{j,i}[B_j] = B_i$, *where* $R|_{i,j}$ *is the projection of* $R$ *to coordinates* $i, j$

*Then a tuple* $\mathbf{a} = (a_1, \ldots, a_n) \in B_1 \times \cdots \times B_n$ *belongs to* $R$ *whenever* $a_K \in R_K$ *for every* $\sim$*-class* $K$.

# A note on binary constraints

Using (Hell, Rafiey or Kazda) and (SD($\wedge$) $\Rightarrow$ BW):

**Theorem**

*Let $\mathbb{A}$ be a conservative relational structure (i.e. containing all unary relations) with at most binary relations.*
*If* Pol($\mathbb{A}$) *is Taylor then* $\mathrm{CSP}(\mathbb{A})$ *has width* $(2, 3)$.

# A conversation

CS guy: Hi, I have this conservative tractable relational structure $\mathbb{A}$. Give me the P-time algorithm for solving CSP over $\mathbb{A}$!

me: Hi, first you have to give me a list of all absorbing subuniverses of all subalgebras of $\mathrm{Pol}(\mathbb{A})$.

CS guy: ??????????? ok, how do I find them?

me: I don't know. I don't know whether it's decidable that a given set is an absorbing subuniverse of $\mathrm{Pol}(\mathbb{A})$ for a given set $\mathbb{A}$ of relations on $A$ (or of a given algebra)...

CS guy: So you proved that a P-time algorithm exists without providing the algorithm????

me: Yes.

CS guy: I don't like it. And I don't like you.

me: I love it. And I don't like you too.

CS guy: See you.

me: See you.

# Decidability of absorption

## Problem

*Is the following problem decidable? Input is a finite algebra $\mathbf{A}$ and a subset $B$. Question is whether $B \triangleleft \mathbf{A}$.*

Affirmative answer would generalize Maróti's result that NU is decidable

Special cases: $|B| = 1$, $\mathbf{A}$ is conservative, $\mathbf{A}$ is Taylor, $\mathbf{A}$ is SD($\wedge$), $\mathbf{A}$ is Mal'tsev

## Problem

*Is the following problem decidable? Input is a finite relational structure $\mathbb{A}$ and a subset $B$. Question is whether $B \triangleleft \mathrm{Pol}(\mathbb{A})$.*

Affirmative answer would generalize the result that NU is decidable

Recent progress: decidable in $SD(\wedge)$ case (see Jakub Bulín's talk)

# More problems

Dichotomy holds for any $\mathbf{B}$ in HSP of a conservative algebra…

**Problem**

*Characterize algebras which are in* $\mathsf{HSP}(\mathbf{A})$ *for some* $\mathbf{A}$ *conservative.*

Recall that for a binary conservative relational structure $\mathbb{A}$,
$\mathsf{Pol}(\mathbb{A})$ is Taylor $\Rightarrow$ $\mathsf{Pol}(\mathbb{A})$ is $\mathsf{SD}(\wedge)$...

Also if $\mathbb{A}$ contains a single binary relation, then
$\mathsf{Pol}(\mathbb{A})$ has Mal'tsev $\Rightarrow$ $\mathsf{Pol}(\Gamma)$ has majority (Kazda + ?)

**Problem**

*Take two important properties $Prop_1$, $Prop_2$ of finite algebras (like omitting types, Mal'tsev, FS...).*
*Is it true that for every conservative relational structure* $\mathbb{A}$ *with (i) at most binary relations (ii) at most one binary relation,*
$\mathsf{Pol}(\mathbb{A})$ *has* $Prop_1$ $\Rightarrow$ $\mathsf{Pol}(\mathbb{A})$ *has* $Prop_2$*?*