The number of homomorphisms into finite algebras

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THE QUESTION

- Fix a set A of size 16

  \# mappings $X \longrightarrow A$ $= 16^n$ ... exponential

  set of size n

- Fix a group $\underline{A}$ of size 16

  \# homomorphisms $\underline{X} \longrightarrow \underline{A}$ $\leq n^4$ ... polynomial

  group of size n
  algebra of size n

  = # ways $\underline{A}$ is "contained in $\underline{X}$" as a quotient

- **Question:** For which algebras $\underline{A}$ is # homomorphisms $\underline{X} \to \underline{A} \leq \text{poly}(|X|)$?

  finite
  finite

OUTLINE

- algebras, homomorphisms
- origin of the question
- examples + ideas
- answer

INTERRUPT!

ALGEBRAS, HOMOMORPHISMS

- algebra = universe + operations on the universe

  e.g. $\underline{A} = (A;\ \bullet^{\underline{A}},\ i^{\underline{A}},\ 1^{\underline{A}})$

  $\bullet^{\underline{A}} : A \times A \longrightarrow A$ binary operation on A

  $i^{\underline{A}} : A \longrightarrow A$ unary

  $1^{\underline{A}} \in A$ constant

  $\underline{X} = (X;\ \bullet^{\underline{X}},\ i^{\underline{X}},\ 1^{\underline{X}})$ similar algebra

- homomorphism $f : \underline{X} \to \underline{A}$ = mapping $X \to A$ preserving operations

$$f(x \bullet^{\underline{X}} y) = f(x) \bullet^{\underline{A}} f(y)$$
$$f(i^{\underline{X}}(x)) = i^{\underline{A}}(f(x))$$
$$f(1^{\underline{X}}) = 1^{\underline{A}}$$

THE ORIGIN

- CSP over $\mathbb{A}$ ... relational structure

  INPUT: $\mathbb{X}$ ... similar structure

  QUESTION: $\exists$ homomorphism $\mathbb{X} \to \mathbb{A}$?

- covers many computational problems (e.g. 3SAT, 3COLORING)
- complexity classification $<^{P}_{NP}$ [Bulatov'17; Zhuk'17]
- method: algebra

- CSP over $\underline{A}$ ... algebra

  INPUT: $\underline{X}$ ... similar algebra

  QUESTION: $\exists$ homomorphism $\underline{X} \to \underline{A}$?

- covers more computational problems
- complexity classification open
- "often": in P for simple reason

  ... we can list all homomorphisms in polynomial time

  $\rightsquigarrow$ the question

QUESTION: For which $\underline{A}$

$$\forall \underline{X} \quad \# \underline{X} \to \underline{A} \leq \mathrm{poly}(|X|) \ ?$$

EXAMPLE : GROUPS

- $\underline{A} = (A;\ \cdot^{\underline{A}},\ i^{\underline{A}},\ 1^{\underline{A}})$ group

$$a\cdot(b\cdot c) = (a\cdot b)\cdot c \qquad \forall a,b,c \in A$$
$$a\cdot 1 = a = 1\cdot a$$
$$a\cdot i(a) = 1 = i(a)\cdot a$$

- $\underline{X} = (X;\ \cdot^{\underline{X}},\ i^{\underline{X}},\ 1^{\underline{X}})$ group / general $\rightsquigarrow$ can enforce identities true in $\underline{A}$

$\boxed{\text{EXAMPLE: NOR}}$

- $\underline{A} = (\{0,1\}; \cdot^{\underline{A}})$

$\cdot^{\underline{A}}$	0	1
0	1	0
1	0	0

consider $a * b := (x \cdot y) \cdot ((x \cdot x) \cdot (y \cdot y))$

what is $a *^{\underline{A}} b$?

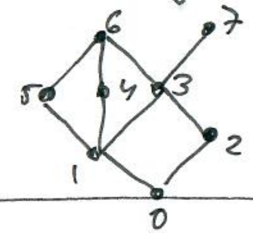
$\rightsquigarrow$ can add term operations

EXAMPLE : SEMILATTICE

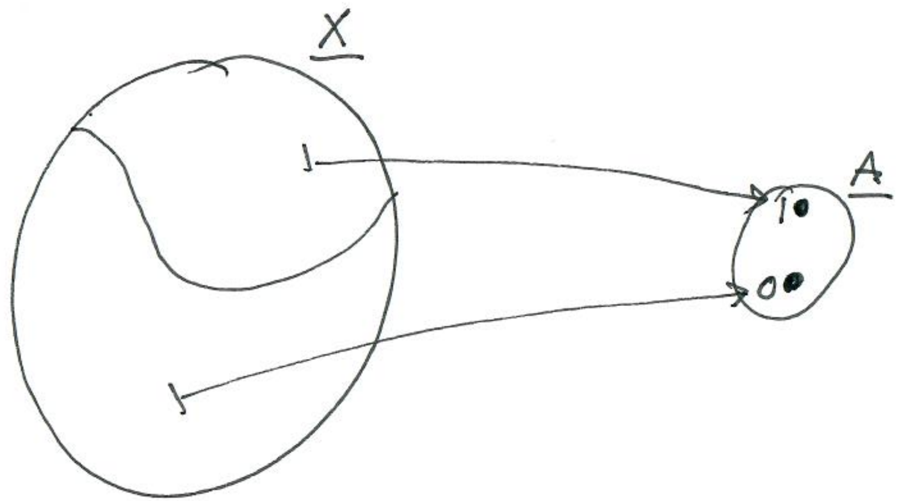
- $\underline{A} = (\{0,1\}; \bullet^{\underline{A}})$ $\quad a \bullet^{\underline{A}} b := \min(a,b)$

semilattice $\underline{S} = (S, \bullet^{\underline{S}})$ $\quad s \bullet^{\underline{S}} r = \inf(s,r)$ w.r.t. some ordering on S

or $\begin{cases} s(rt) = (sr)t \\ ss = s \\ sr = rs \end{cases}$



- WLOG $\underline{X}$ is a semilattice
- consider $f: \underline{X} \to \underline{A}$



$\boxed{\text{EXAMPLE : MAJORITY}}$

- $\underline{A} = (\{0,1\};\ m^{\underline{A}})$ $\qquad m^{\underline{A}}(0,0,0) = m^{\underline{A}}(1,0,0) = m^{\underline{A}}(0,1,0) = m^{\underline{A}}(0,0,1) = 0$
  $\qquad m^{\underline{A}}(1,1,1) = m^{\underline{A}}(0,1,1) = m^{\underline{A}}(1,0,1) = m^{\underline{A}}(1,1,0) = 1$

- consider $f : \underline{X} \to \underline{A}$ — not onto / onto

  - if $f(\mathbf{s}) = 0$ then f preserves $\qquad s^{\underline{A}}(x,y) := m^{\underline{A}}(0,x,y) \quad x,y \in A$
    $\qquad s^{\underline{X}}(x,y) := m^{\underline{X}}(\mathbf{s},x,y) \quad x,y \in X$

    such an s exists

  - $s^{\underline{A}}$ is min!

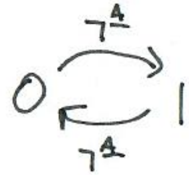
$\rightsquigarrow$ 0 can add constants to A

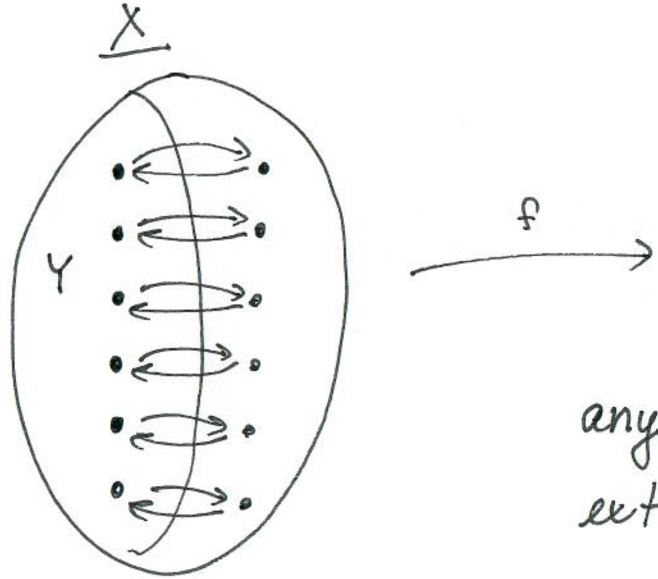
EXAMPLE: UNARY

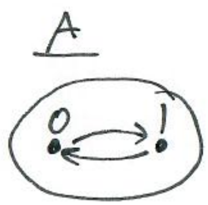
not polynomial

- $\underline{A} = (\{0,1\}; \neg^{\underline{A}})$

a	0	1
$\neg^{\underline{A}}$	1	0







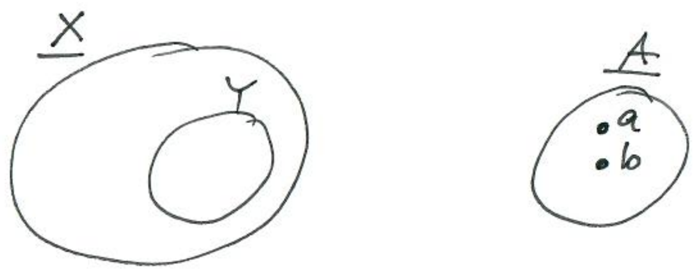
any mapping $Y \to \{0,1\}$
extends to homomorphism

- $\bigcirc$s, these examples, Post $\to$ sufficient to answer the question for $|A|=2$
  $|A|=2$, contains only essentially unary operations ... exponentially many homomorphisms
  $|A|=2$, doesn't — " — ... polynomially many
- for $|A|>2$, the Tame Congruence Theory useful [Hobby, McKenzie '88 + ...]

MANY HOMOMORPHISMS

- For some $\underline{A}$ there exists arbitrarily large $\underline{X}$ such that
  - $\exists Y \subseteq X$ large (eg. $|Y| \geq |X|^{\frac{1}{37}}$)
  - any mapping $Y \to \{a, b\}$ extends to a homomorphism $\underline{X} \to \underline{A}$

    some specific elements of A



- Then $\# \underline{X} \to \underline{A} \not\leq \mathrm{poly}(|X|)$

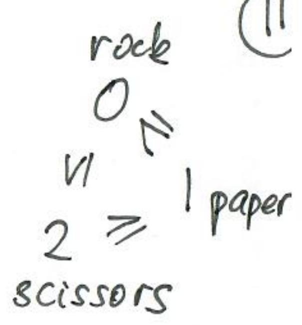
- Turns out: otherwise $\# \underline{X} \to \underline{A} \leq \mathrm{poly}(|X|)$!

EXAMPLE: ROCK-PAPER-SCISSORS

polynomial

$a \cdot^{A} b$ = looser

$0 \cdot 1 = 1 \cdot 0 = 0 = 0 \cdot 0$
$1 \cdot 2 = 2 \cdot 1 = 1 = 1 \cdot 1$
$0 \cdot 2 = 2 \cdot 0 = 2 = 2 \cdot 2$



$\underline{A} = (\{0,1,2\}; \cdot^{A}, 0^{A}, 1^{A}, 2^{A})$

- take $\underline{X}$, $f: \underline{X} \to \underline{A}$
- consider $p(x) = x.0$

a	0	1	2
$p^{A}(a)$	0	0	2

- $Z := p^{\underline{X}}(X)$ $\overset{!!!}{\Rightarrow}$ $f(Z) \subseteq \{0,2\}$

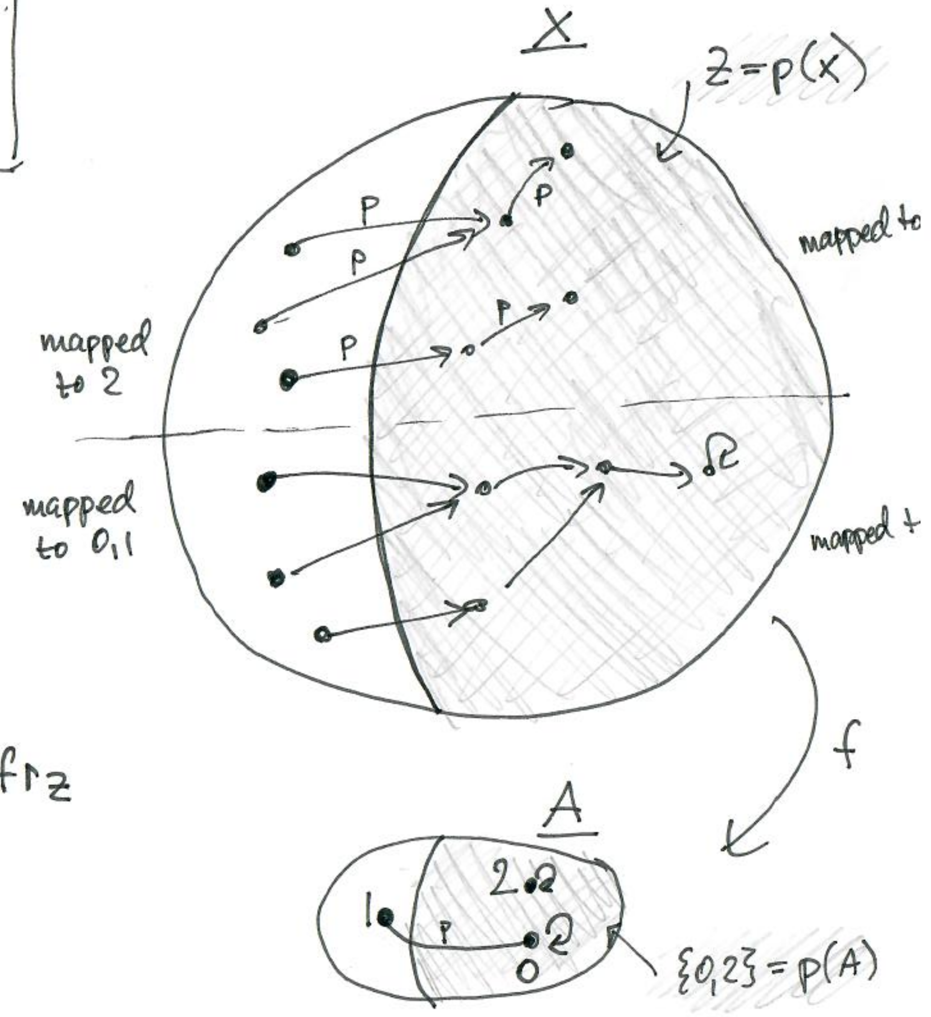
  $\rightsquigarrow$ polynomially many $f \restriction Z$

- Fix $f \restriction Z$ for $x \in X$

  if $f \restriction Z\,(p(x)) = 2$ then $f(x) = 2$

  if $ = 0$ then $f(x) \in \{0,1\}$

  $\rightsquigarrow$ polynomially many extensions of $f \restriction Z$



THE ANSWER

THEOREM: For a finite algebra $\underline{A}$ the following are equivalent.

(1) $\# \underline{X} \to \underline{A} \leq \mathrm{poly}(|X|)$

(2) no subalgebra of $\underline{A}$ has a nonzero strongly abelian congruence.

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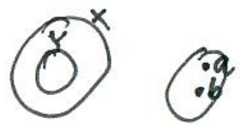
(2) • depends on term operations of $\underline{A}$

• can be tested in P (given $\underline{A}$ on input)

• if (2) then, given $\underline{X}$, all homomorphisms $\underline{X} \to \underline{A}$ can be listed in P

if $\neg(2)$ • there exist $\underline{X}$'s such that $\# \underline{X} \to \underline{A} \geq 2^{|X|^{\frac{1}{k}}}$ constant depending on $\underline{A}$

for the simple reason 

• complexity of CSP over $\underline{A}$ still open in general

THANK YOU!