Constraint Satisfaction Problems
Part I: Complexity, Logic, Algebra

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CoCoSym: Symmetry in Computational Complexity

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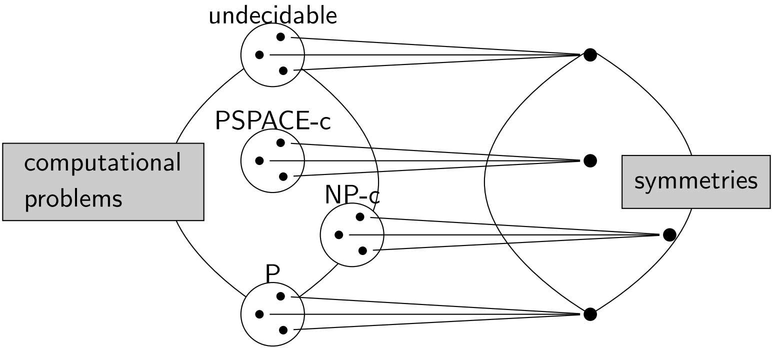
. . . there will also be three more focused topics, providing examples of (. . .) interactions between logic, algebra, and complexity

- Part I: logic, algebra, and complexity in **Constraint Satisfaction Problems (CSPs)** over fixed finite templates
- Part II: analysis, probability, and topology in (a variant of) CSP

In CSP: complexity captured by symmetry

$$\mathrm{CoolFunc}\colon \text{computational problems} \longrightarrow \text{objects capturing symmetry}$$
$$\text{kernel of } \mathrm{CoolFunc} \;=\; \text{polynomial time reducibility}$$



CSPs over fixed finite templates

- tiny portion of problems on the left
- kernel $\subsetneq$ polynomial time reducibility

CSP

Fix $\mathbb{A} = (A; R, S, \dots)$ relational structure

Definition ($\mathrm{CSP}(\mathbb{A})$)

Input: pp-sentence ϕ, eg. $(\exists x_1 \exists x_2 \dots) R(x_1, x_3) \wedge S(x_5, x_2) \wedge \dots$
Answer Yes: ϕ satisfied in $\mathbb{A}$
Answer No: ϕ not satisfied in $\mathbb{A}$

Search version: Find a satisfying assignment.
Search looks harder, but it's not [Bulatov, Jeavons, Krokhin'05]

Fact: Always in NP.

$\mathbb{K}_3 = (A; R)$ where

- $A = \{\textit{lilac}, \textit{mauve}, \textit{cyclamen}\}$
- $R =$ (binary) inequality relation on A

Input of $\mathrm{CSP}(\mathbb{K}_3)$ is, e.g.

$(\exists x_1 \exists x_2 \dots \exists x_4) R(x_1, x_2) \wedge R(x_1, x_3) \wedge R(x_1, x_4) \wedge R(x_2, x_3) \wedge R(x_2, x_4)$

Viewpoint

- variables = vertices
- clauses (constraints) = edges

$\mathrm{CSP}(\mathbb{K}_3)$ is the 3-coloring problem for graphs

Fact: It is NP-hard (7-coloring NP-hard, 2-coloring in P)

- $3\mathbb{NAE}_2 = (\{0,1\}; 3NAE_2)$ where
  $3NAE_2 = \text{all but } \{(0,0,0),(1,1,1)\}$
  $\mathrm{CSP}(3\mathbb{NAE}_2) =$ positive not-all-equal 3-SAT
  $=$ 2-coloring problem for 3-uniform hypergraphs

- $3\mathbb{NAE}_4 = (\{0,1,2,3\}; 3NAE_4)$, where $3NAE_4$ still ternary
  $\mathrm{CSP}(3\mathbb{NAE}_4) =$ 4-coloring problem for 3-uniform hypergraphs

- $1\mathbb{IN}3 = (\{0,1\}; 1in3)$ where
  $1in3 = \{(0,0,1),(0,1,0),(1,0,0)\}$
  $\mathrm{CSP}(1\mathbb{IN}3) =$ positive 1-in-3 SAT

Fact: All NP-hard

$3\mathbb{LIN}_5 = (\mathbb{Z}_5; L_{0000}, L_{0001}, \dots, L_{4444})$ where e.g.

$L_{1234} = \{(x, y, z) \in \mathbb{Z}_5^3 : 1x + 2y + 3z = 4\}$

(note: relations are affine subspaces of $\mathbb{Z}_5^3$)

$\mathrm{CSP}(3\mathbb{LIN}_5)$ = solving systems of linear equations in $\mathbb{Z}_5$

Fact: In P

Symmetry

polymorphism of $\mathbb{A}$: mapping $f : A^n \to A$
compatible with every relation

compatible with R: f applied component-wise to tuples in R
is a tuple in R

Example: $f(x_1, \ldots, x_4) = 2x_1 + 3x_2 + 3x_3 + 3x_4$ $\quad f : \mathbb{Z}_5^4 \to \mathbb{Z}_5$
is compatible with each L_{abcd}
because $f(\mathbf{v}_1, \ldots, \mathbf{v}_4)$ is an affine combination of these vectors (as $2 + 3 + 3 + 3 = 1$)
and L_{abcd} is an affine subspace

$\mathrm{Pol}(\mathbb{A})$: the set of all polymorphisms (it is a "clone")
$=$ set of (multivariable) symmetries of $\mathbb{A}$

Jeavons'98: On the algebraic structure of combinatorial problems

Theorem

Complexity of $\mathrm{CSP}(\mathbb{A})$ *is determined by* Pol($\mathbb{A}$)*:*
If Pol($\mathbb{A}$) $\subseteq$ Pol($\mathbb{B}$) *then* $\mathrm{CSP}(\mathbb{B})$ *reduces to* $\mathrm{CSP}(\mathbb{A})$*.*

Proof.

If Pol($\mathbb{A}$) $\subseteq$ Pol($\mathbb{B}$), then relations in $\mathbb{B}$ can be defined from relations in $\mathbb{A}$ by a pp-formula.

[Geiger'69, Bondarčuk, Kalužnin, Kotov, Romov'69]

This gives a computational reduction of $\mathrm{CSP}(\mathbb{B})$ to $\mathrm{CSP}(\mathbb{A})$. □

So: $\mathrm{CSP}(3\mathbb{LIN}_5)$ is in P because $3\mathbb{LIN}_5$ has a lot of polymorphs
$\mathrm{CSP}(1\mathbb{IN}3)$ is NP-complete because $1\mathbb{IN}3$ has few

System of functional equations is, e.g.

$$f(g(x,y),z) = g(x,h(y,z))$$
$$m(y,x,x) = m(y,y,y)$$
$$m(x,x,y) = m(y,y,y)$$

Satisfied in $\mathcal{M}$, where $\mathcal{M}$ is a set of functions:
symbols can be interpreted in $\mathcal{M}$ so that
each equality is (universally) satisfied

Example: The above system is satisfied in Pol($3\mathbb{LIN}_5$):

- take $f(x,y) = g(x,y) = h(x,y) = x$
  (note: projections are always polymoprhisms)
- take $m(x,y,z) = x - y + z$

Bulatov, Jeavons, Krokhin'05: Classifying the complexity of constraints using finite algebras + Bodirsky'08: PhD thesis

Theorem

Complexity of $\mathrm{CSP}(\mathbb{A})$ *is determined by*
systems of functional equations satisfied in Pol($\mathbb{A}$):
If each system satisfied in Pol($\mathbb{A}$) *is satisfied in* Pol($\mathbb{B}$),
then $\mathrm{CSP}(\mathbb{B})$ *reduces to* $\mathrm{CSP}(\mathbb{A})$.

Proof.

Previous theorem, pp-definitions $\rightarrow$ pp-interpretations,
the HSP theorem [Birkhoff'35] □

So: $\mathrm{CSP}(3\mathbb{LIN}_5)$ is in P because
Pol($3\mathbb{LIN}_5$) satisfies strong systems of functional equations.

Barto, Opršal, Pinsker'18: The wonderland of reflections

minor condition = system of functional equations, each of the form

symbol(*variables*) = *symbol*(*variables*),

e.g. $m(y,x,x) = m(y,y,y)$, $m(x,x,y) = m(y,y,y)$

Theorem

Complexity of $\mathrm{CSP}(\mathbb{A})$ *determined by*

minor conditions satisfied in $\mathrm{Pol}(\mathbb{A})$*:*

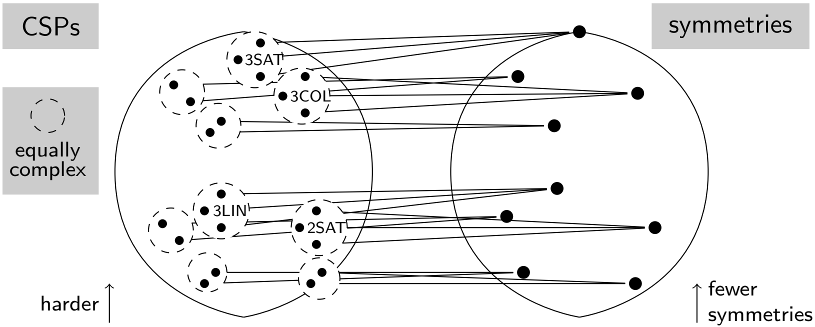
If each minor condition satisfied in $\mathrm{Pol}(\mathbb{A})$ *is satisfied in* $\mathrm{Pol}(\mathbb{B})$*,*

then $\mathrm{CSP}(\mathbb{B})$ *reduces to* $\mathrm{CSP}(\mathbb{A})$*.*

Proof.

pp-interpretation $\to$ pp-construction,

version of the HSP theorem. □

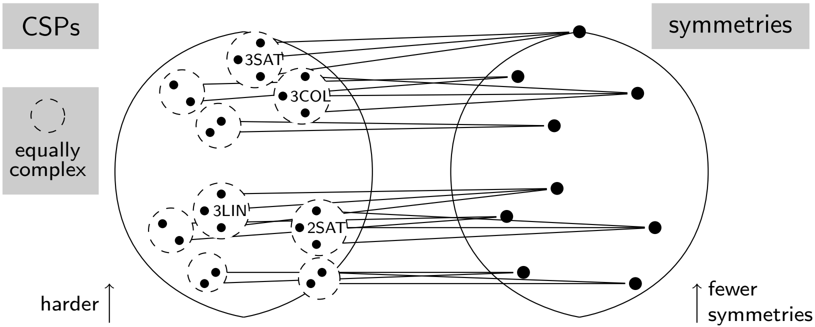


(1) polymorphisms

(2) systems of functional equations satisfied by polymorphisms

(3) minor conditions satisfied by polymorphisms

Where are the borderlines between complexity classes?



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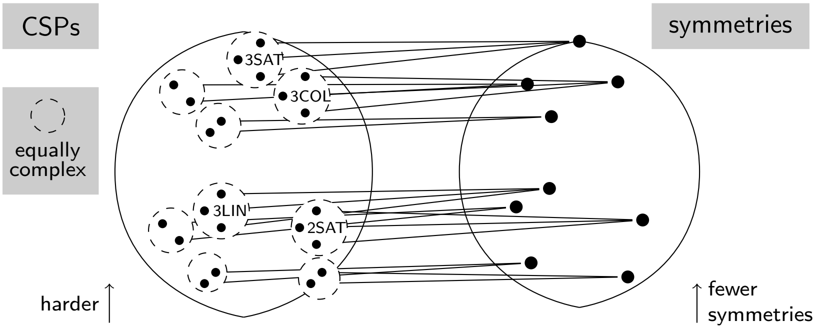

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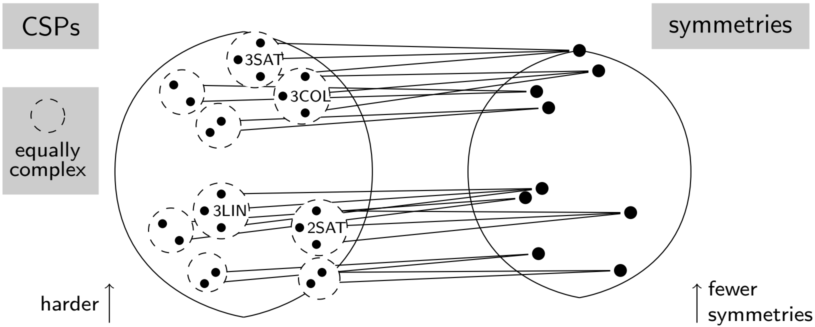

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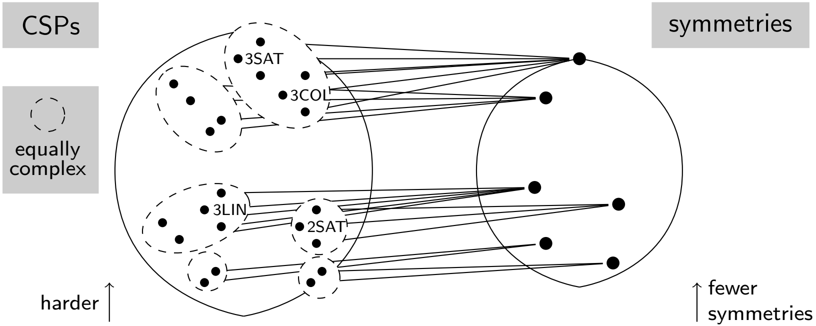

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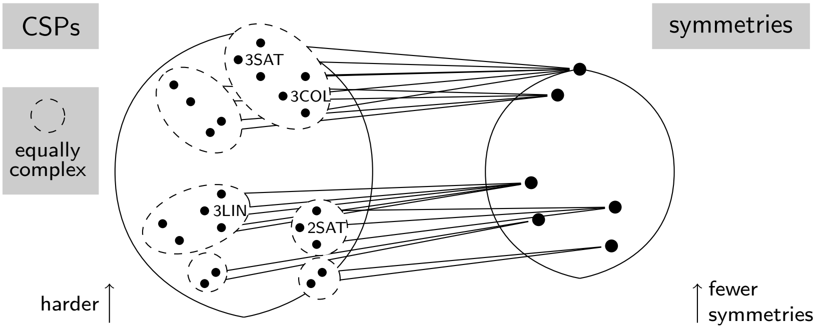

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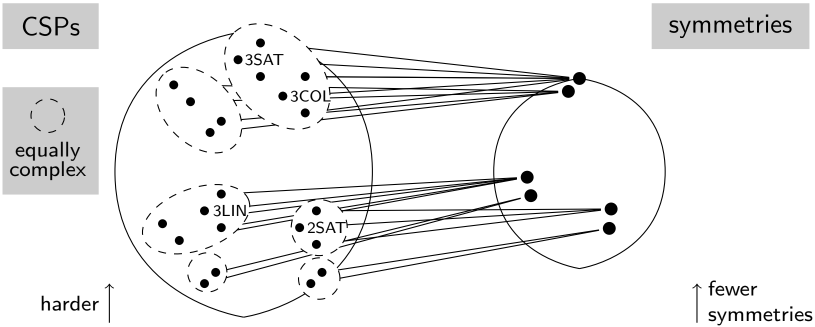

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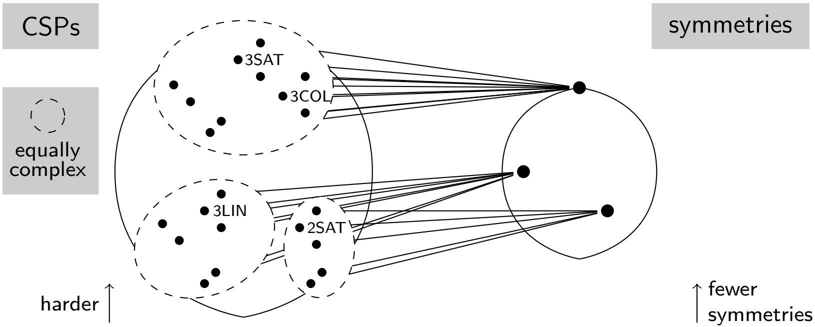


(1) polymorphisms

(2) systems of functional equations satisfied by polymorphisms

(3) minor conditions satisfied by polymorphisms

Where are the borderlines between complexity classes?

Minor condition is trivial:

satisfied in every $\mathrm{Pol}(\mathbb{A})$

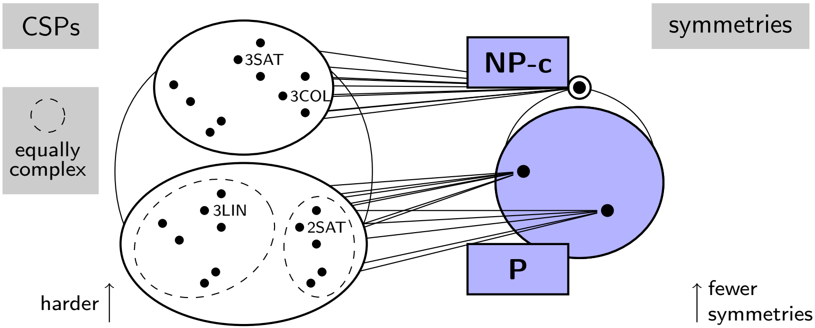
$=$ satisfied in $\mathcal{P}$, the set of projections on $\{0, 1\}$

Corollary

If $\mathrm{Pol}(\mathbb{A})$ *satisfies only trivial minor conditions,*
then $\mathrm{CSP}(\mathbb{A})$ *is NP-hard.*

Theorem ([Bulatov'17], [Zhuk'17])

If $\mathrm{Pol}(\mathbb{A})$ *satisfies some non-trivial minor condition,*
then $\mathrm{CSP}(\mathbb{A})$ *is in P.*



- only trivial minor conditions ⇒ NP-complete
- some nontrivial minor condition ⇒ P

Further steps?

(Barto,) Bulín, Krokhin, Opršal: Algebraic approach to promise constraint satisfaction

Definition ($\mathrm{MinorCond}(N, \mathcal{M})$)

Input: minor condition **X** with symbols of arity N
Answer Yes: **X** is trivial (=satisfied in $\mathcal{P}$)
Answer No: **X** not satisfied in $\mathcal{M}$

Theorem

Let $\mathcal{M} = \mathrm{Pol}(\mathbb{A})$*. The following computational problems are equivalent for a large enough* N*.*

(i) $\mathrm{CSP}(\mathbb{A})$
(ii) $\mathrm{MinorCond}(N, \mathcal{M})$

Consequence: 3rd step
Proof: direct, simple, known

Proof 1: Reduction from CSP

Given input of $\mathrm{CSP}(3\mathbb{NAE}_2)$, eg.

$$(\exists a, b, c, d)\ R(c, a, b) \wedge R(a, d, c)$$

transform it to a minor condition, eg.

$$\begin{aligned}
f_1(x_1, x_0, x_0, x_0, x_1, x_1) &= g_c(x_0, x_1)\\
f_1(x_0, x_1, x_0, x_1, x_0, x_1) &= g_a(x_0, x_1)\\
f_1(x_0, x_0, x_1, x_1, x_1, x_0) &= g_b(x_0, x_1)
\end{aligned}$$

$$\begin{aligned}
f_2(x_1, x_0, x_0, x_0, x_1, x_1) &= g_a(x_0, x_1)\\
f_2(x_0, x_1, x_0, x_1, x_0, x_1) &= g_d(x_0, x_1)\\
f_2(x_0, x_0, x_1, x_1, x_1, x_0) &= g_c(x_0, x_1)
\end{aligned}$$

"Yes input $\to$ Yes input": easy

"No input $\to$ No input": for contrapositive use $y \mapsto g_y(0, 1)$.

Given a minor condition, e.g.

$$f(x_1, x_2, x_1, x_3) = g(x_1, x_2, x_3)$$
$$h(x_3, x_1) = g(x_1, x_2, x_3)$$

- introduce variables f_{a_1,a_2,a_3,a_4} one for each $(a_1, \ldots, a_4) \in A^4$, h_{a_1,a_2}, and g_{a_1,a_2,a_3}.
- so evaluation of f's $\leftrightarrow$ function $f : A^4 \to A$
- express that f, g, h are polymorphisms (by constraints)
- merge variables to enforce the equations

The proof only uses bipartite minor conditions:

- Two disjoint set of symbols LHS, RHS.
- Each equation of the form

$$f(variables) = g(x_1, x_2, \ldots, x_N)$$

where $f \in LHS$ and $g \in RHS$

Remarks

How to show that $\mathcal{M} = \mathrm{Pol}(\mathbb{A})$ satisfies only trivial minor conditions?

Theorem

The following are equivalent

- *$\mathcal{M}$ satisfies only trivial minor conditions*
- *There is a mapping $\xi : \mathcal{M} \to \mathbb{N}$*
  - *if f is of arity n, then $\xi(f) \in \{1, 2, \dots, n\}$*
    *(**think**: an important coordinate of f)*
  - *ξ behaves nicely with minors, eg. if*

    $$f(x_3, x_2, x_1, x_2, x_2, x_1) = g(x_1, x_2, x_3)$$

    and $\xi(f) = 5$, then $\xi(g) = 2$.

How to devise algorithms if $\mathcal{M} = \mathrm{Pol}(\mathbb{A})$ satisfies some nontrivial minor condition?

Theorem

The following are equivalent.

- $\mathcal{M}$ *satisfies some nontrivial minor condition*
- $\mathcal{M}$ *satisfies, for some $n \geq 2$, the minor condition*

$$c(x_1, x_2, \ldots, x_n) = c(x_2, \ldots, x_n, x_1)$$

  [Barto, Kozik'12]

- *. . .*
- *. . . zillion other characterizations . . .*
- *. . .*

General problem: Given a structure $\mathfrak{A}$ and 1st order sentence ϕ (the same language), decide whether $\mathfrak{A}$ satisfies ϕ.

CSP

- fix a finite relational structure
- restrict to primitive positive (pp-) sentences

Another problem: Given a structure $\mathfrak{A}$ and 1st order sentence ϕ (different language), decide whether symbols in ϕ can be interpreted in $\mathfrak{A}$ so that $\mathfrak{A}$ satisfies ϕ.

Our case: solving functional equations over an algebra

- fix a finite algebraic structure
- restrict to universally quantified conjunction of (special) equations
- take a promise version