Constraint Satisfaction Problems
Part II: Analysis, Probability, Topology

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CoCoSym: Symmetry in Computational Complexity

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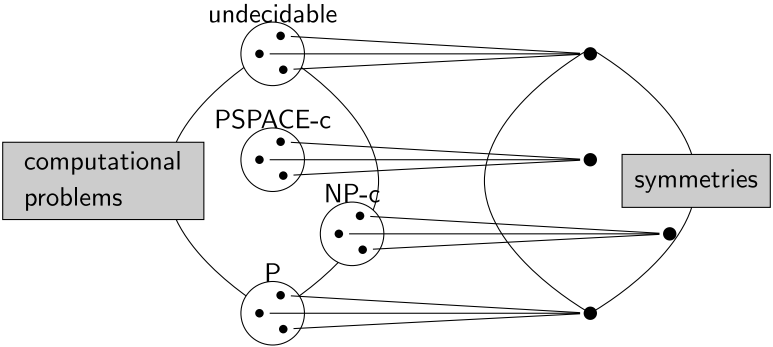
$\mathrm{CoolFunc}$: computational problems $\longrightarrow$ objects capturing symmetry
kernel of $\mathrm{CoolFunc}$ $=$ polynomial time reducibility



- we "almost" have it for CSPs but
  kernel $\subsetneq$ polynomial time reducibility
- $\mathrm{CSP}(\mathbb{A})$ is equivalent to $\mathrm{MinorCond}(N, \mathrm{Pol}(\mathbb{A}))$

$\mathrm{CSP}(\mathbb{A})$ is often NP-complete

What can we do?

1. Try to satisfy only some fraction of the constraints, eg.

   for a satisfiable 3SAT instance,
   find an assignment satisfying at least 90% of the clauses

2. Try to satisfy a relaxed version of all constraints, eg.

   for a 3-colorable graph,
   find a 37-coloring

Approximation

satisfying a fraction of constraints

3SAT is hard to approximate

Theorem (Håstad'01)

The following problem is NP-complete for every $\epsilon > 0$
Input: *3SAT instance, eg.* $(x_1 \vee \neg x_4 \vee x_3) \wedge (\neg x_2 \vee x_5 \vee \neg x_3) \wedge \ldots$
Answer Yes: *it is satisfiable*
Answer No: *no* $(7/8 + \epsilon)$*–fraction of clauses is satisfiable*

Corollary: It is NP-hard to satisfy 90% of clauses
of a satisfiable 3SAT instance.

Proof.

Reduction from a version of the Label Cover problem
(known to be NP-hard).

Uses Fourier analysis of Boolean functions. □

$\mathrm{LabelCover}(N)$ is $\mathrm{CSP}([N]; \langle M_\phi \rangle_\phi)$, where

- $[N] = \{1, 2, \ldots, N\}$
- for each $\phi : [N] \to [N]$ we have a relation
  $$M_\phi = \{(a, \phi(a)) : a \in [N]\}$$

Additionally

- we have two disjoint sets of variables L, R
- each constraint $M_\phi(x, y)$ has $x \in L$, $y \in R$

Example

$(\exists x_1, \ldots, y_1, \ldots)\ M_\phi(x_3, y_1) \wedge M_{\phi'}(x_2, y_4) \wedge M_{\phi''}(x_5, y_1) \wedge \ldots$

Definition ($\mathrm{GapLabelCover}(N, \epsilon)$)

Input: like $\mathrm{LabelCover}(N)$, eg. $\phi = M_\phi(x_3, y_1) \wedge M_{\phi'}(x_2, y_4) \wedge \dots$
Answer Yes: ϕ is satisfiable
Answer No: no ϵ–fraction of constraints is satisfiable

Theorem

For every $\epsilon > 0$ *there exists* N *such that* $\mathrm{GapLabelCover}(N, \epsilon)$ *is NP–complete*

Proof.

The PCP theorem [Arora, Lund, Motwani, Sudan, Szegedy'98]

Parallel Repetition Thoerem [Raz'98] □

The following two problems are the same!

- (bipartite) $\mathrm{MinorCond}(N, \mathcal{P})$ ie. deciding whether a given minor condition is trivial
- $\mathrm{LabelCover}(N)$ ie. deciding whether a given label cover input is satisfiable

Because:

- interpretation of f and g by projections making the following equation true
  $f(x_3, x_1, x_1, x_2, x_1) = g(x_1, x_2, x_3, x_4, x_5)$
- corresponds to a satisfying assignment of $M_\phi(f, g)$ where
  $\phi : 1 \mapsto 3, \quad 2, 3, 5 \mapsto 1, \quad 4 \mapsto 2$
- under the correspondence
  $i \leftrightarrow$ projection onto the ith coordinate

Remark: often implicitely used ("long code")

A formulation of the Gap Label Cover

$\mathrm{GapLabelCover}(N, \epsilon)$

Input: bipartite minor condition (symbols of arity N)

Answer Yes: it is trivial

Answer No: no ϵ–fraction of equations is trivial

PCSP

satisfying a relaxed version of all constraints

Fix 2 relational structures in the same language

- $\mathbb{A} = (A; R^{\mathbb{A}}, S^{\mathbb{A}}, \dots)$
- $\mathbb{B} = (B; R^{\mathbb{B}}, S^{\mathbb{B}}, \dots)$
- there is a homomorphism $\mathbb{A} \to \mathbb{B}$ (eg. $A \subseteq B, R^{\mathbb{A}} \subseteq R^{\mathbb{B}}, \dots$)

Definition ($\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$)

Input: pp-sentence ϕ, eg. $(\exists x_1 \exists x_2 \dots) R(x_1, x_3) \wedge S(x_5, x_2) \wedge \dots$
Answer Yes: ϕ satisfied in $\mathbb{A}$
Answer No: ϕ not satisfied in $\mathbb{B}$

Search version: Find a $\mathbb{B}$-satisfying assignment given an $\mathbb{A}$-satisfiable input.

(it may be a harder problem, we don't know)

Recall: $\mathbb{K}_n = (\{1, 2, \ldots, n\}; \text{inequality})$

$\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_4)$

Input: a graph
Answer Yes: it is 3-colorable
Answer No: it is not 4-colorable

Search version: Find a 4-coloring of a 3-colorable graph

Fun facts:

- **Theorem:** it is NP-hard [Brakensiek, Guruswami'16]
  (more generally $\mathrm{PCSP}(\mathbb{K}_n, \mathbb{K}_{2n-2})$ is NP-hard)
- $\mathrm{PCSP}(\mathbb{K}_n, \mathbb{K}_{2n-1})$ [Bulín, Krokhin, Opršal'19]
- 6-coloring 3-colorable graph: complexity not known
- **Conjecture**: k-coloring, l-colorable graph always NP-hard
  ($k \geq l \geq 3$)

Recall: $3\mathbb{NAE}_k$ ternary not-all-equal relation on a k-element set

$\mathrm{PCSP}(3\mathbb{NAE}_2, 3\mathbb{NAE}_{137})$

Input: a 3-uniform hypergraph
Answer Yes: it is 2-colorable
Answer No: it is not 137-colorable

Theorem: It is NP-hard [Dinur,Regev,Smyth'05]
(more generally $\mathrm{PCSP}(3\mathbb{NAE}_l, 3\mathbb{NAE}_k)$ NP-hard for every $k \geq l \geq 2$)

Recall: $1\mathbb{IN}3 = (\{0,1\};\{(1,0,0),(0,1,0),(0,0,1)\})$

$\mathrm{PCSP}(1\mathbb{IN}3, 3\mathbb{NAE}_2)$

Input: a 3-uniform hypergraph
Answer Yes: there is a 2-coloring such that exactly one vertex in each hyperedge receives 1
Answer No: it is not 2-colorable

Fact: It is in P. Algorithm for finding a 2-coloring of a Yes input:

- for each hyperedge $\{x, y, z\}$ write $x + y + z = 1$
- solve the system over $\mathbb{Q} \setminus \{\frac{1}{3}\}$ (it is solvable in $\{0, 1\}$)
- assign $x \mapsto 1$ iff $x > 1/3$

Note: algorithm uses infinite domain CSP
Theorem: infinity is necessary [Barto'19]

polymorphism of $(\mathbb{A}, \mathbb{B})$: mapping $f : A^n \to B$
compatible with every relation-pair

compatible with $(R^{\mathbb{A}}, R^{\mathbb{B}})$: f applied to tuples in $R^{\mathbb{A}}$
is a tuple in $R^{\mathbb{B}}$

Example: $f(x_1, \dots, x_{97}) = 1$ iff $\frac{\sum x_i}{97} > \frac{1}{3}$ $\quad f : \{0,1\}^{97} \to \{0,1\}$
is compatible with $(1in3, 3NAE_2)$

$\mathrm{Pol}(\mathbb{A}, \mathbb{B})$: the set of all polymorphisms (it is a "minion")
= set of (multivariable) symmetries of $(\mathbb{A}, \mathbb{B})$

1st step (polymorphisms):
can be generalized [Brakensiek, Guruswami'18]
using [Pippenger'02]

2nd step (systems of functional equations):
makes no sense
since polymorphisms can no longer be composed

3rd and 4th step (minor conditions): the same as CSP!

Theorem ([Bulín, Krokhin, Opršal'19])

Let $\mathcal{M} = \mathrm{Pol}(\mathbb{A}, \mathbb{B})$*. The following computational problems are equivalent for a large enough* N*.*

(i) $\mathrm{CSP}(\mathbb{A}, \mathbb{B})$
(ii) $\mathrm{MinorCond}(N, \mathcal{M})$

Hardness of PCSPs

Theorem ([Brakensiek, Guruswami'16])

$\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_4)$ *is NP-complete*

Proof.

- enough to show that $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_4)$ satisfies only trivial minor conditions.
- equivalently, there is a mapping $\xi : \mathcal{M} \to \mathbb{N}$
  - if f is of arity n, then $\xi(f) \in \{1, 2, \dots, n\}$ (**think**: an important coordinate of f)
  - ξ behaves nicely with minors
- for every $f \in \mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_4)$ of arity n
  - there exists $t \in \{1, 2, 3, 4\}$ (a trash color)
  - and there exists $i \in \{1, 2, \dots, n\} =: \xi(f)$ and α such that
  - $f(x_1, \dots, x_n) = \alpha(x_i)$ whenever $f(x_1, \dots, x_n) \neq t$

□

Theorem ([Bulín, Krokhin, Opršal'19])

$\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$ *is NP-complete*

Proof.

- the previous criterion is not applicable
- there is a mapping $\xi : \mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5) \to \mathrm{Pol}(3\mathbb{NAE}_2, 3\mathbb{NAE}_{enough})$ that behaves nicely with minors
- **Remark:** such a $\xi : \mathcal{M} \to \mathrm{Pol}(3\mathbb{NAE}_2, 3\mathbb{NAE}_{enough})$ exists iff $\mathcal{M}$ does not satisfy $t(y,x,x,x,y,y) = t(x,y,x,y,x,y) = t(x,x,y,y,y,x)$
- so every minor condition satisfied in $\mathrm{Pol}(\mathbb{K}_3, \mathbb{K}_5)$ is satisfied in $\mathrm{Pol}(3\mathbb{NAE}_2, 3\mathbb{NAE}_{enough})$
- so $\mathrm{PCSP}(3\mathbb{NAE}_2, 3\mathbb{NAE}_{enough})$ reduces to $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$

□

Theorem

Let $\mathcal{M} = \mathrm{Pol}(\mathbb{A}, \mathbb{B})$*. If there exists* $C \in \mathbb{N}$ *and a mapping* $\xi : \mathcal{M} \to P(\mathbb{N})$ *such that*

- *if* f *is of arity* n*, then* $\xi(f) \subseteq \{1, 2, \ldots, n\}$, $|\xi(f)| \leq C$ *(**think**: a small set of important coordinates of* f*)*
- ξ *behaves nicely with minors, eg. if*

$$f(x_3, x_2, x_1, x_2, x_2, x_1) = g(x_1, x_2, x_3)$$

  and $\xi(f) = \{4, 5, 6\}$*, then* $\xi(g) \cap \{1, 2\} \neq \emptyset$

Then $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ *is NP-complete.*

- we have $\xi : \mathcal{M} \to P(\mathbb{N})$, want to show that
  (a) $\mathrm{GapLabelCover}(N, 1/C^2)$ reduces to
  (b) $\mathrm{MinorCond}(N, \mathcal{M})$ (via trivial reduction)
- **Recall:**
  - **Input:** bipartite minor condition (symbols of arity N)
  - **Answer Yes:** it is trivial
  - **Answer No:**
    (a) no $1/C^2$–fraction of equations is trivial
    (b) not satisfied in $\mathcal{M}$
- "Yes input → Yes input": trivial
- "No input → No input": for contrapositive:
  - take a valid interpretation in $\mathcal{M}$
  - reinterpret f as the i-th projection, where $i \in \xi(f)$ random
  - each equation is satisfied with probability $\geq 1/C^2$
  - so expected fraction of satisfied equations is $\geq 1/C^2$
  - so some $1/C^2$-fraction is trivial

Theorem ([Dinur,Regev,Smyth'05])

$\mathrm{PCSP}(3\mathbb{NAE}_2, 3\mathbb{NAE}_{137})$ *is NP-complete.*

Proof.

- Let $f \in \mathrm{Pol}(3\mathbb{NAE}_2, 3\mathbb{NAE}_{137})$ of arity n
- **Crucial claim**: there exists a set $I =: \xi(f)$ of coordinates and $c \in [137]$ such that
  - $|I| < 200$
  - $f(whatever, \underbrace{1, 1, \dots, 1}_{I}, whatever) \neq c$
- **Enough to show**: there are two dijoint set J, K of coordinates such that
  - $|J| = |K| = (n - 200)/2$
  - $f(0, \dots, 0, \underbrace{1, \dots, 1}_{J}, 0, \dots 0) = f(0, \dots, 0, \underbrace{1, \dots, 1}_{K}, 0, \dots 0)$
- since f is a polymorphism

Theorem ([Dinur,Regev,Smyth'05])

$\mathrm{PCSP}(3\mathbb{NAE}_2, 3\mathbb{NAE}_{137})$ *is NP-complete.*

Proof.

- Assume the converse: whenever J, K of size $(n-200)/2$ are disjoint, then
$$f(0,\ldots,0,\underbrace{1,\ldots,1}_{J},0,\ldots 0) \neq f(0,\ldots,0,\underbrace{1,\ldots,1}_{K},0,\ldots 0)$$
- This gives us a 137-coloring of the following graph
  - vertices: subsets of $[n]$ of size $(n-200)/2$
  - J and K adjacent iff thery are disjoint
- Such a coloring does not exist! [Lovász'78]
- Proof uses algebraic topology, started Topological Combinatorics

□

- **Warning:** The presented sketch of proof does not quite work
- **Often:** Important coordinates of functions are determined by analytical (counting) properties
- **Here:** Based on topological properties
  - close in spirit to the (deeper parts of) CSP theory
  - other example where this works: $\mathrm{PCSP}(\mathbb{C}_{137}, \mathbb{K}_3)$ [Krokhin,Opršal]
  - this is the way to go, because

geometry $>$ counting

Summary

CSP

- = a version of the LabelCover (and MinorCond) problem
- Complexity captured by a piece of information about polymorphisms

PCSP is cool and fun

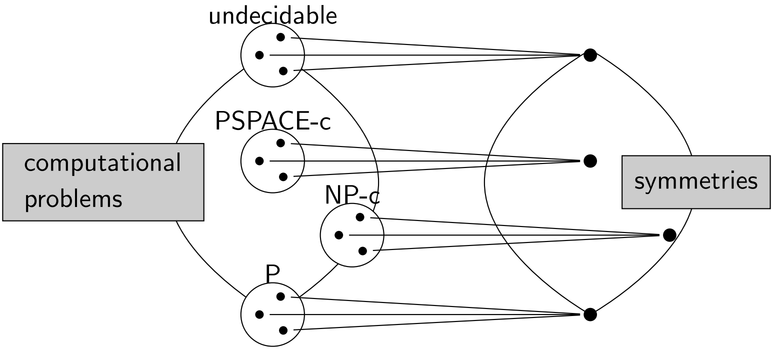
- Basics work but a lot is open: eg. borderlines
- More algorithms needed
- More interesting hardness proofs (PCP, topology)
- **Q:** What else can we forget about polymorphisms?

Reading

- Barto, Krokhin, Willard: Polymorphisms, and How to Use Them
- other surveys in this Dagstuhl Follow-Up volume
- Barto, Bulín, Krokhin, Opršal: Algebraic Approach to Promise Constraint Satisfaction (coming soon)

$\mathrm{CoolFunc}$: computational problems $\longrightarrow$ objects capturing symmetry

kernel of $\mathrm{CoolFunc}$ $=$ polynomial time reducibility



Thank you!