The Dichotomy for Conservative Constraint Satisfaction Problems Revisited

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Fixed template CSPs

Γ ... template ... fixed set of relations on a finite set (domain) A

Definition ($\mathrm{CSP}(\Gamma)$ - Constraint Satisfaction Problem over Γ)

INPUT: Formula of the form

$$(x_1, x_2) \in R_1 \;\; \& \;\; (x_3, x_1, x_3, x_4) \in R_2 \;\; \& \;\; x_7 \in R_3 \;\; \& \ldots$$

where each R_i is in Γ (R_1 binary, R_4 4-ary, R_3 unary)
(i.e. a conjunction of atomic formulas over Γ)

QUESTION: Is the formula satisfiable?

Examples: Various forms of SAT, (Di)graph reachability, Equations over ...

Alternative formulation (if Γ is finite): the homomorphism problem with a fixed target relational structure

The dichotomy conjecture

Conjecture (Feder, Vardi 93, generalized version)

For every Γ, $\mathrm{CSP}(\Gamma)$ is tractable or NP-complete.

Recent (2000 –) highlights:

(0) It is a universal algebraic problem Bulatov, Jeavons, Krokhin

(1) Conjecture is true when $|A| \leq 3$ Bulatov

(2) Conjecture is true if Γ contains all unary relations on A (so called conservative CSPs) Bulatov

(3) Applicability of "Gaussian elimination like" methods characterized Dalmau, Bulatov, Berman, Idziak, Marković, McKenzie, Valeriote, Willard

(4) Applicability of local consistency methods characterized Barto, Kozik

(5) A couple of nice tricks Maróti

Good times, bad times Led Zeppelin

(1) Conjecture is true when $|A| \leq 3$

(2) Conjecture is true for conservative templates

- Proofs use heavy universal algebraic machinery
  ($\Rightarrow$ hard to understand for a non-specialist)
- Long and complicated
  ($\Rightarrow$ hard to understand for a specialist)
- Techniques very specific for the problem

(3) Applicability of "Gaussian elimination like" methods

(4) Applicability of local consistency methods

- Proofs don't use any heavy machinery
- Bring new general notions and results, applicable elsewhere

To move on we need to understand (1),(2) better.

Good times, bad times Led Zeppelin

(1) Conjecture is true when $|A| \leq 3$

(2) Conjecture is true for conservative templates

(3) Applicability of "Gaussian elimination like" methods

(4) Applicability of local consistency methods

(5) A couple of nice tricks

Fortunately (1),(2) are consequences of (3),(4),(5):

(1') Conjecture is true when $|A| \leq 3$ (4?) Marković et al

(2') Conjecture is true for conservative templates Barto

Also...

(4') Applicability of local consistency methods Bulatov

- Using similar techniques as original proofs of (1) and (2)

Polymorphisms

polymorphism of Γ . . . an operation on A compatible with all relations in Γ

Theorem (Bulatov, Jeavons, Krokhin)

If Γ has no "nice" polymorphims, then $\mathrm{CSP}(\Gamma)$ is NP-complete

Where "nice" for core Γ = e.g. cyclic... $t(x, \ldots, x) = x$, $t(x_1, x_2, \ldots, x_n) = t(x_2, \ldots, x_n, x_1)$ Barto, Kozik

Conjecture (Bulatov, Jeavons, Krokhin)

If Γ has a "nice" polymorphism, then $\mathrm{CSP}(\Gamma)$ is tractable.

Similar conjectures for finer complexity classification.

The algorithm

Theorem

If Γ is a conservative template which has a "nice" polymorphism, then $\mathrm{CSP}(\Gamma)$ is tractable.

Proof: Algorithm for domains of size $k \to$ alg for doms of size $k+1$ (simplified, but not too much):

(Step 1) Transform the instance to an equivalent instance which is consistent enough

(Step 2) Find a small restriction which is still consistent enough (4)

(Step 3) Use the algorithm for smaller domains (to certain restricted instances). Either we find a solution, or we can delete some elements and repeat, or

(Step 4) If you cannot delete anything, use (3)

Step 2 - Finding small, consistent enough restriction

Let Γ be a fixed conservative template (on the domain A).

Definition

Let $C \subseteq A$.

A subset $B \subseteq C$ is an *absorbing subuniverse* of C, if there exists a polymorphism t of Γ such that
$t(a_1, \ldots, a_n) \in B$
whenever all a_i's are in C and
all a_i's but at most one are in B

- Start with a proper absorbing subuniverse
- Walk until you stabilize
- Restrict
- Repeat

A conversation

CS guy: Hi, I have this conservative tractable template Γ. Give me the P-time algorithm for solving CSP over Γ!

me: Hi, first you have to give me a list of all absorbing subuniverses of all subsets of A.

CS guy: ??????????? ok, how do I find them?

me: I don't know. I don't know whether it's decidable that a given set is an absorbing subuniverse of A for a given set Γ of relations on A (or of a given algebra)...

CS guy: So you proved that a P-time algorithm exists without providing the algorithm????

me: Yes.

CS guy: I don't like it.

me: I love it.

CS guy: See you.

me: See you.

A note on binary constraints

Using (Hell, Rafiey or Kazda) and ((2) or (4)):

Theorem

If Γ *is conservative and contains only at most binary relations, then* $\mathrm{CSP}(\Gamma)$ *is solvable by local consistency methods, or NP-complete.*

Thank you!!!