UNIVERSAL ALGEBRA

tutorial

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MFO Workshop „Homogeneous Structures: Model Theory meets Universal Algebra"

What is UA?

- model theory without relations
- (semi)group theory for functions of arity ≥ 1

Achievements

- good understanding of 2-element algebras
- some understanding of bigger algebras

$\rightsquigarrow$
- direct decomposability
- decidability of 1st order theory
- dualizibility
- polynomially many models
- finite axiomatization
- dichotomy theorem for finite domain CSPs
  [Bulatov '17] [Zhuk '17]

Tools

- basic: ....., ..., free algebras, ..., ...
- commutator theory 70s [Smith, ...]
- tame congruence theory 80s [McKenzie, Hobby, ...]
- Bulatov's theory 00s [Bulatov]
- absorption theory 10s [B., Kozik, Zhuk]

quite developed (commutator theory, tame congruence theory)

finite (tame congruence theory, Bulatov's theory, absorption theory)

mess (Bulatov's theory, absorption theory)

This tutorial

- basics, tools
- biased (towards
  - finite domain CSP
  - stuff I know
)

Outline

basics & abstract nonsense

I algebras

II clones

III relational side

IV wonderland of reflections, Taylor clones

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tools

V commutator theory

VI TCT

VII Bulatov's theory

VIII Absorption theory

I ALGEBRAS as models

signature $\Sigma = \{f, g, \dots\}$ arity: $\Sigma \to \mathbb{N}_0$

algebra $\underline{A} = (A; f^{\underline{A}}, g^{\underline{A}}, \dots)$ $f^{\underline{A}}: A^{\mathrm{arity}(f)} \to A$ (total!)

universe

basic operations

basic constructions

(S) for $B \subseteq A$ $\underline{B} := (B; f^{\underline{A}}\restriction_{B^{\cdots}}, \dots)$ — restrictions — subalgebra

- makes sense iff B is a subuniverse of $\underline{A}$, written $B \leq \underline{A}$
  = invariant under all operations
- $S(\underline{A}) :=$ all subalgebras of $\underline{A}$

(P) for any X $\underline{A}^X := (A^X;$ component-wise$)$ — power

- subpower = invariant relation … $R \leq \underline{A}^X$
  $\mathrm{Inv}_\infty \underline{A} := \{R \leq \underline{A}^X;\ X \text{ a set}\}$
  $\mathrm{Inv}\,\underline{A} := \{R \leq \underline{A}^n;\ n \in \mathbb{N}\}$

(H) for equivalence $\sim$ on A $\underline{A}/_\sim := (A/\sim;$ representatives$)$ — quotient

- makes sense iff $\sim$ is a congruence of $\underline{A}$, i.e. $\sim \leq \underline{A}^2$
- quotient $\doteq$ homomorphic image

identity

= pair of terms (s,t) over $\{x_1,\dots,x_m\}$, written $s\approx t$

- $\underline{A}$ satisfies $s\approx t$ if $s^{\underline{A}} = t^{\underline{A}}$ term operations

important subpower

- consider $\underline{A}^{A^n}$ for some $n\in\mathbb{N}$
  - elements = n-ary operations
  - $f^{\underline{A}^{A^n}}(g_1,\dots,g_r) = f(g_1,\dots,g_r)$ where "generalized composition"

    $f(g_1,\dots,g_r)(a_1,\dots,a_n) = f(g_1(a_1,\dots,a_n),\dots,g_r(a_1,\dots,a_n))$
- take $F :=$ n-ary term operations of $\underline{A} \subseteq A^{A^n}$
  - $F \leq \underline{A}^{A^n}$ important subpower (= invariant relation)
  - corresponding $\underline{F}$ = free algebra for $\underline{A}$ over n generators

HSP theorem [Birkhoff 30s]

one reason we like identities

$\underline{B} \in \dots HSPPSPHPS(\underline{A}) \iff \underline{B} \in HSP(\underline{A})$

$\iff \underline{B}$ satisfies all identities satisfied by $\underline{A}$

Proof: last $\Leftarrow$

- for simplicity $\underline{B} = (\{1,2,\dots,n\}; \dots)$
- take $\underline{F} \in SP(\underline{A})$ above, define $F \longrightarrow B$, $t^{\underline{A}} \longmapsto t^{\underline{B}}(1,2,\dots,n)$
  - well defined from the assumptions
  - homomorphism $\underline{F} \to \underline{B}$, onto
- so $\underline{B} \in HSP(\underline{A})$

II CLONES

signature-free UA

Often: If $\underline{A} = (A; \dots)$, $\underline{B} = (A; \dots)$ have the same term operations, they can be considered "equal"

e.g. invariant relations (congruences, ...) are the same

$\boxed{\text{Clone on } A}$

= set of operations of arity ≥ 1 closed under forming term operations

= set of operations — " — containg projections π_i^n and closed under $f(g_1, \dots, g_r)$

- permutation group → transformation monoid → clone

e.g. Aut(...) End(...) Pol(..)

$\boxed{\text{How to specify a clone}}$

- $\mathrm{Clo}(\underline{A}) :=$ all term operations of $\underline{A}$
  - $\mathrm{Clo}_n(\underline{A}) =$ n-ary members of $\mathrm{Clo}(\underline{A}) = F$ from 4
  - analogue of specifying permutation group by generators
- $\mathrm{Pol}(A; \underbrace{R_1, \dots}_{\text{relations}}) := \{ f ;\ \forall i\ R_i \leq (A; f) \} =$ homomorphisms from powers
  - "always" possible :

$\boxed{\text{Theorem}}$ [Geiger; Bodnarčuk, Kalužnin, Kotov, Romov 60s]

$$\mathrm{Clo}(\underline{A}) = \mathrm{Pol}(\mathrm{Inv}_\infty \underline{A}) \text{ for finite } A = \mathrm{Pol}(\mathrm{Inv}\, \underline{A})$$

Proof: $\supseteq$

- take n-ary $f \in$ RHS
- $\mathrm{Clo}_n \underline{A}$ is an invariant relation, so f preserves it
- $f = f(\underset{\substack{\Uparrow \\ \mathrm{Clo}_n A}}{\pi_1^n}, \dots, \underset{\Uparrow}{\pi_n^n}) \in \mathrm{Clo}_n \underline{A}$

Examples

- projections • all operations
- all idempotent operations $= \mathrm{Pol}(A; (\{a\})_{a\in A})$

  $f(x, \dots, x) = x$ trivial unary part
- all essentially unary functions (or some es.—.)

  trivial > unary parts
- $\mathrm{Clo}(\{0,1\}; \wedge)$, $\mathrm{Clo}(\{0,1\}; \vee)$

  binary minimum maximum
- $\mathrm{Clo}(\{0,1\}; \wedge, \vee) = \mathrm{Pol}(\{0,1\}; \leq, \{0\}, \{1\})$
- $\mathrm{Clo}(\{0,1\}; \mathrm{maj}) = \mathrm{Pol}(\{0,1\}; \leq, \neq)$

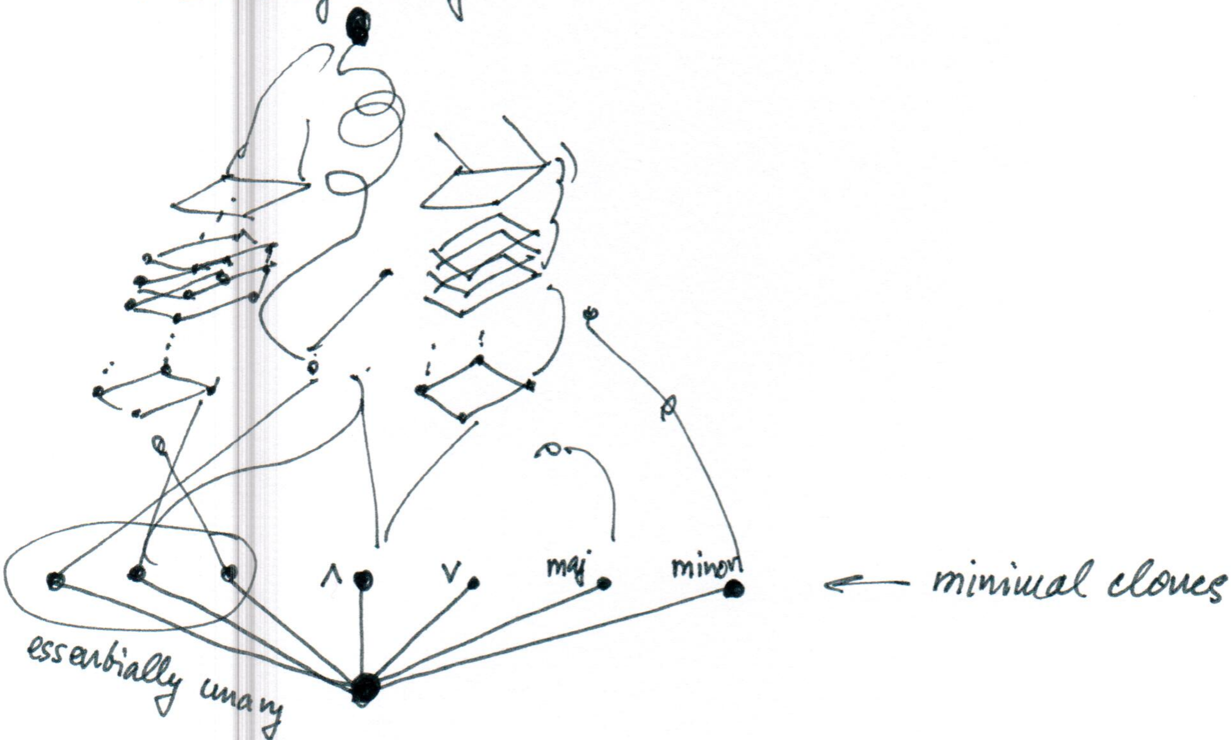
  $\mathrm{maj}(x, y, z) =$ majority in x, y, z
- $\mathrm{Clo}(\{0,1\}; \mathrm{minor})$

  $\min(x, y, z) =$ minority in case of 1 exception

  $= x + y + z \mod 2$
- $\mathrm{Clo}(\text{vector space}) =$ linear forms $= \mathrm{Pol}(\text{subspaces})$

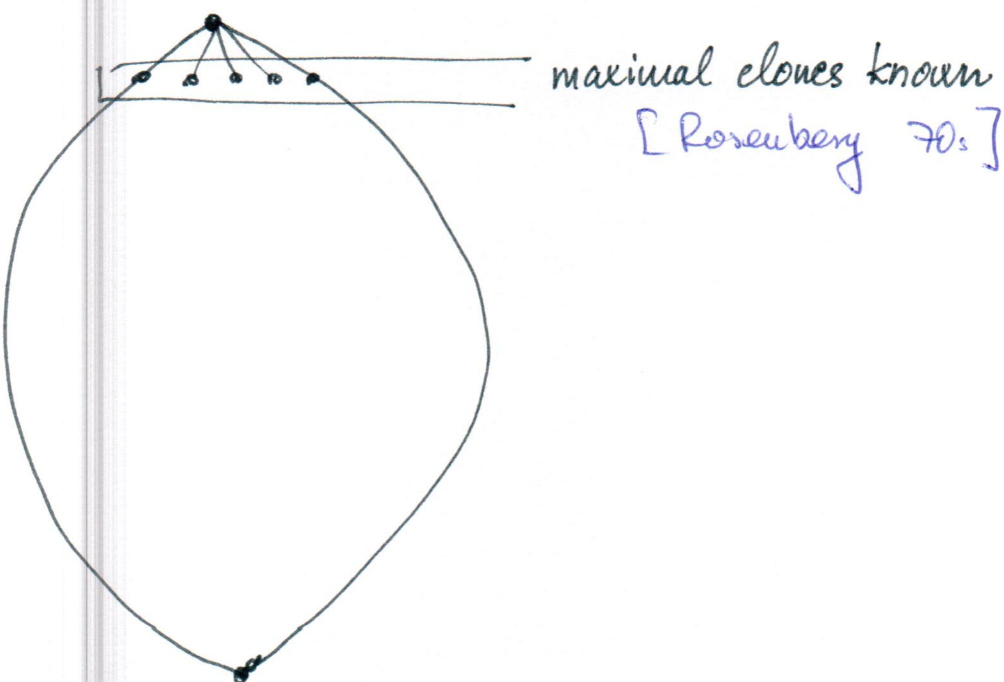
Clones on {0,1}

- all known – Post's lattice [Post ?s]
- countably many



Clones on finite A>2

- continuum many



basic constructions

- S, P, H (quotients) work, invariant rel.
- expansion $\mathcal{C} \in E(\mathcal{D})$ if $\mathcal{C} \supseteq \mathcal{D}$
- homomorphisms
  - concrete, not so important
  - abstract, analogue of group homomorphisms

clone homomorphisms

= mapping $\xi : \mathcal{C} \to \mathcal{D}$ that

- preserves arities
- preserves terms $\xi(t^{\mathcal{C}}) = \xi(t)^{\mathcal{D}}$

= • preserves projections $\xi(\pi^n_i) = \pi^n_i$

  • preserves composition $\xi(f(g_1,\dots)) = \xi(f)(\xi(g_1),\dots\dots)$

= preserves identities and arities

e.g. associative binary operation in $\mathcal{C}$ is mapped to — " — in $\mathcal{D}$

- depends only on the "abstract clone"

$\boxed{\text{• HSP theorem: } \mathcal{D} \in EHSP(\mathcal{C}) \iff \exists \text{ homo } \mathcal{C} \to \mathcal{D}}$

comparison

unary version (bijective)	$\geq$ unary version: signature free	$\geq$ unary version: with signatures
permutation group	clone	algebra
group	abstract clone	equational class
group homos	clone homos	interpretations
group actions	clone actions	algebras in eq. cl.
Cayley representation	free algebras	— " —

$\boxed{\text{equational condition}}$ * terminology?

= condition for a clone $\mathcal{C}$ of the form

$$\underbrace{\exists f_1 \in \mathcal{C}\ \exists f_2 \in \mathcal{C} \ \dots\dots}_{\text{possibly } \infty} \quad \underbrace{\text{conjunction of identities for } f_1, \dots}_{\text{possibly } \infty}$$

e.g. $\exists f \in \mathcal{C}_2\ \exists g \in \mathcal{C}_3 \quad f(g(z,x,x),y) \approx g(y,x,x)$

- If $\mathcal{C}$ satisfies condition M & $\exists$ homo $\mathcal{C} \to \mathcal{D}$, then $\mathcal{D}$ satisfies M
- related to "Mal'tsev conditions" in UA
- M is <u>trivial</u> if it is satisfied in every clone $\iff$ in the clone of projections (on ≥ 2 elements)
- $\mathcal{C}$ is <u>equationally trivial</u> if it satisfies only trivial equational conditions
- strong equational conditions $\leftrightarrow$ nice properties of invariant relations, e.g.:

$\boxed{\text{Theorem}}$ $\mathcal{A}$ clone on A [Mal'tsev Jónsson]

(i) $\exists m \in \mathcal{A}_3 \quad m(x,x,y) \approx y \approx m(y,x,x)$ ← Mal'tsev operation

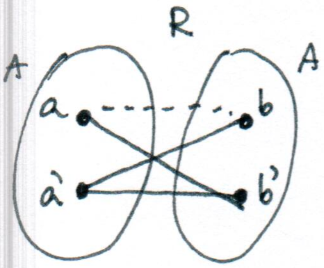
(ii) $\forall R \in \mathrm{Inv}_2 \mathcal{B}$, where $\exists$ homo $\mathcal{A} \to \mathcal{B}$ is rectangular

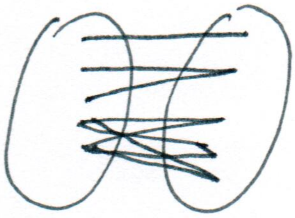
e.g. $\mathcal{A} = (\{0,1\}; \text{minor})$

recall $\mathcal{B} \in EHSP(\mathcal{A})$
so this translates to properties of relations in $\mathrm{Inv}_\infty \mathcal{A}$

Rectangularity

- $R \subseteq A \times A$ <u>rectangular</u> if $\forall a, a', b, b' \in A$ $\begin{array}{l} ab' \in R \\ a'b' \in R \\ a'b \in R \end{array} \Rightarrow ab \in R$





Theorem $\mathcal{A}$ clone @ [Mal'tsev 50s]

(i) $\exists m \in \mathcal{A}_3$ $m(xxy) \approx y \approx m(y, x, x)$

(ii) $\forall R \in \mathrm{Inv}_2 \mathcal{B}$, where $\exists$ homo $\mathcal{A} \to \mathcal{B}$, is rectangular

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Proof (i) $\Rightarrow$ (ii)

- $\mathcal{B}$ has m such that $m(xxy) \approx y \approx m(yxx)$
- $m \begin{pmatrix} a & a' & a' \\ b' & b' & b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{array}{cccc} \Cap & \Cap & \Cap & \Uparrow \\ R & R & R & R \end{array}$$

(ii) $\Rightarrow$ (i)

- take $\mathcal{B} \in SP(\mathcal{A})$ the clone on $\mathcal{A}_2$
- take R the smallest invariant relation containg

$$\begin{pmatrix} \pi^2_1 \\ \pi^2_2 \end{pmatrix}, \begin{pmatrix} \pi^2_2 \\ \pi^2_2 \end{pmatrix}, \begin{pmatrix} \pi^2_2 \\ \pi^2_1 \end{pmatrix} \qquad \text{"}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} y \\ y \end{pmatrix}, \begin{pmatrix} y \\ x \end{pmatrix}\text{"}$$

$$R = \left\{ \begin{pmatrix} f(\pi^2_1, \pi^2_2, \pi^2_2) \\ f(\pi^2_2, \pi^2_2, \pi^2_1) \end{pmatrix} ; f \in \mathcal{A}_3 \right\} = \left\{ \text{"}\begin{pmatrix} f(xyy) \\ f(yyx) \end{pmatrix}\text{"} ; f \in \mathcal{A}_3 \right\}$$

- it contains $\begin{pmatrix} x \\ x \end{pmatrix}$
- the witnessing f ~~is~~ satisfies $\begin{array}{l} f(xyy) \approx x \\ f(yyx) \approx x \end{array}$

2-decomposability

- $R \subseteq A^n$ is 2-decomposable if it is determined by projections on pairs of coordinates, i.e

$$(a_1, \dots, a_n) \in R \iff \forall i,j \in n \quad (?, ?, \dots, \underset{\substack{\uparrow\\ i}}{a_i}, ?, \dots, ?, \underset{\substack{\uparrow\\ j}}{a_j}, ?, \dots, ?) \in R$$

Theorem $\mathbb{A}$ clone

(i) $\exists m \in \mathbb{A}_3 \quad m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x$

majority operation

(ii) $\forall R \in \mathrm{Inv}_n \mathbb{B}$, where $\exists$ homo $\mathbb{A} \to \mathbb{B}$, is 2-decomposable

---

translates into properties of relations in $\mathrm{Inv}_\circ \mathbb{A}$

recall $(\{0,1\}; \mathrm{maj})$

Proof: $n = 3$ similar

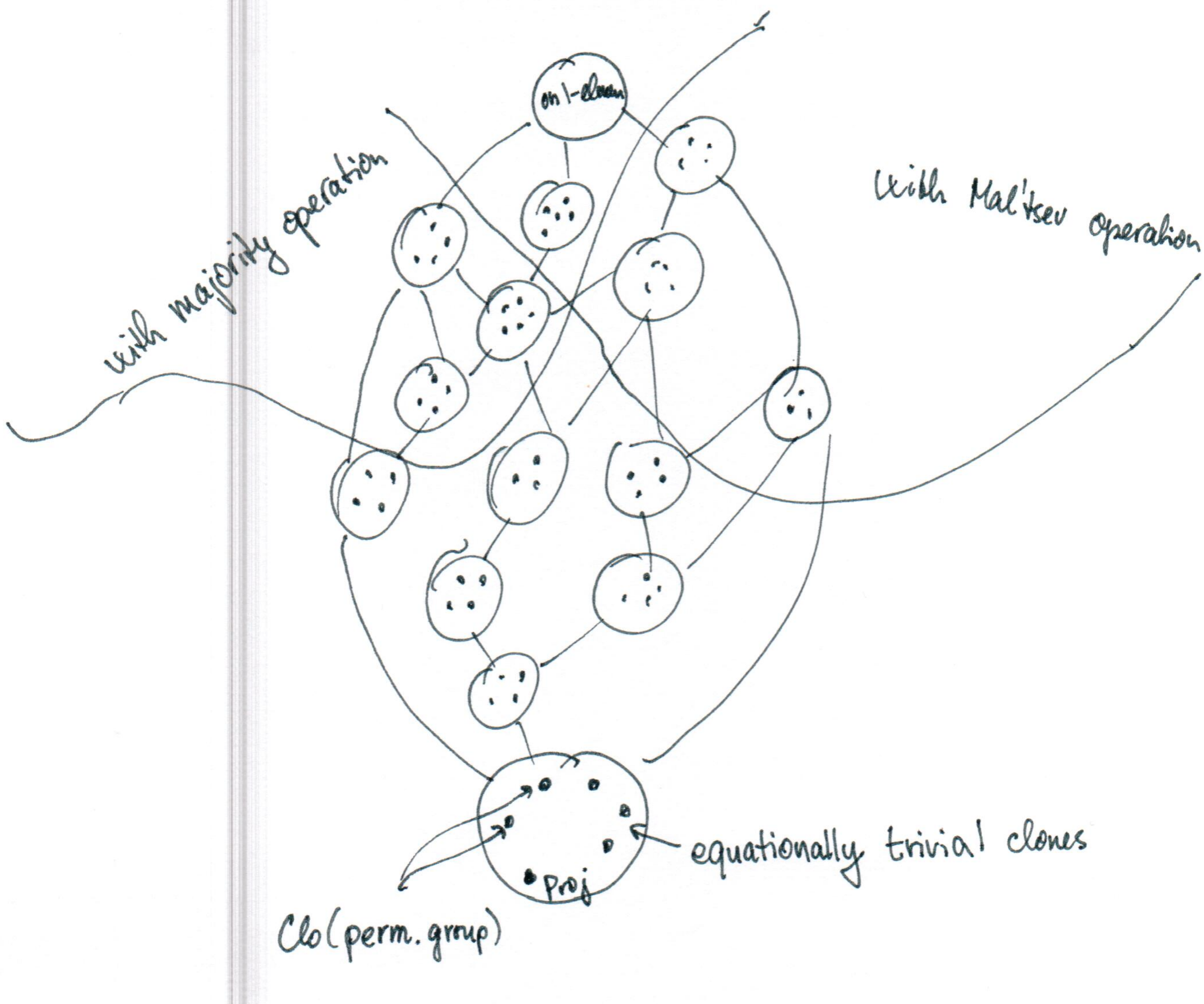
instead of "$\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} y \\ y \end{pmatrix}, \begin{pmatrix} y \\ x \end{pmatrix}$" use "$\begin{pmatrix} x \\ x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \\ x \end{pmatrix}, \begin{pmatrix} y \\ x \\ x \end{pmatrix}$"

(i) $\Rightarrow$ (ii) for $n > 3$ induction

When are 2 algebras essentially equal

- $\underline{A} = \underline{B}$
- $\underline{A} \cong \underline{B}$
- $Clo(\underline{A}) = Clo(\underline{B})$ $\rightsquigarrow$ ordering by $\subseteq$
- $Clo(\underline{A}) \cong Clo(\underline{B})$
- $Clo(\underline{A}) \rightleftarrows Clo(\underline{B})$ $\rightsquigarrow$ ordering by $\exists$homo

homomorphism ordering of clones



III RELATIONAL SIDE

- consider clone $\mathcal{C}$ on A, for convenience A finite
  - recall $\mathcal{C} = \mathrm{Pol}(\mathrm{Inv}\,\mathcal{C})$

  $\leadsto$ instead of $\mathcal{C}$ we can study $\mathrm{Inv}\,\mathcal{C}$
  - what can we do?

    on algebraic side : form term operations

    on relational side : ?.. pp-define

pp-definitions

- $S \subseteq A^n$ pp-definable from $R_1, R_2 \dots$ (relations on A) if it can be defined by a formula that uses

  $R_1, \dots, =, \wedge, \exists$

  e.g. $S(x,z) \stackrel{\text{def}}{\equiv} \exists y \; R_1(x,y) \wedge R_2(y,z)$

  i.e. $S = R_1 \circ R_2$ is pp-definable from R_1, R_2

- relational clone on A = set of relations closed under pp-definable relations (+ containing $\emptyset$)
- $\mathrm{Relclo}(\mathbb{A})$ = all relations pp-definable from $\mathbb{A}$ $(+\emptyset)$

  $\mathbb{A}$: relational structure / set of relations on A
- $\mathrm{Inv}\,\mathcal{C}$ is a relational clone, in fact

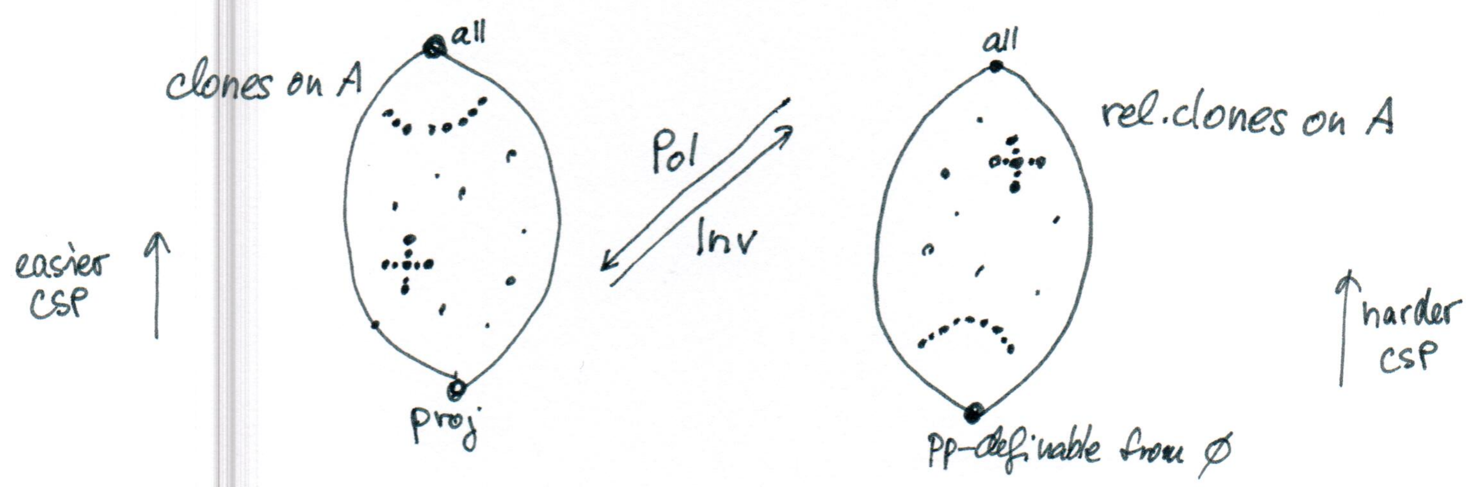
Theorem [6, BKKR 60's]

For $\mathbb{A} = (A; \dots)$ A finite $\quad \mathrm{Relclo}(\mathbb{A}) = \mathrm{Inv}(\mathrm{Pol}\,\mathbb{A})$

Proof: as always..

clones ⟷ relational clones

- for fixed finite A, Pol, Inv are mutually inverse order-reversing bijections between clones and rel. clones



CSP

- $\mathbb{A} = (A; R_1, \ldots, R_k)$ relational structure
- $\mathrm{CSP}(\mathbb{A})$ INPUT : pp-sentence over $\mathbb{A}$
  QUESTION : true?

$\mathbb{A}$ pp-defines $\mathbb{B}$ then $\mathrm{CSP}(\mathbb{B}) \leq \mathrm{CSP}(\mathbb{A})$

$\Updownarrow$

$\mathrm{Relclo}(\mathbb{B}) \subseteq \mathrm{Relclo}(\mathbb{A}) \overset{\text{finite } A}{\Longleftrightarrow} \mathrm{Pol}(\mathbb{B}) \supseteq \mathrm{Pol}(\mathbb{A})$

pp-interpretation

- $\mathbb{A}^{(A; \ldots)}$ **pp-interprets** $\mathbb{B}^{(B; \ldots)}$ if $\exists i : A^n \to B$ "everything pp-def."

$\Updownarrow$ A, B finite

still $\mathrm{CSP}(\mathbb{B}) \leq \mathrm{CSP}(\mathbb{A})$

$\exists$ homo $\mathrm{Pol}(\mathbb{A}) \longrightarrow \mathrm{Pol}(\mathbb{B})$

$\longrightarrow$ $\mathrm{Pol}(\mathbb{A})$ equationally trivial iff $\mathbb{A}$ pp-interprets every finite

IV WONDERLAND OF REFLECTIONS

[B, Opršal, Pinsker '17]

- we know (for finite)

$$\boxed{\mathbb{A} \text{ pp-interprets } \mathbb{B} \Leftrightarrow \mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B}) \Leftrightarrow \mathrm{Pol}(\mathbb{B}) \in EHSP_{fin}(\overset{\mathrm{Pol}\,\mathbb{A}}{\cancel{\mathbb{A}}})}$$

$+ \quad CSP(\mathbb{B}) \leq CSP(\mathbb{A})$

- we can

allow more constructions	weaken requirment on homo	~~allow more constr.~~
⇝ pp-construction	⇝ h1 homomorphism	~~⇝ reflection~~

- while keeping $\Leftrightarrow$ & $CSP(\mathbb{B}) \leq CSP(\mathbb{A})$

pp-construction

- <u>homomorphic equivalence</u> $\mathbb{B} \in He(\mathbb{A})$ if
  $\exists$homos $\mathbb{A} \rightleftarrows \mathbb{B}$ (⇝ same signature)

- <u>pp-power</u> $\mathbb{B} \in$ pp-power$(\mathbb{A})$ if $B = A^k$ and
  and relations in $\mathbb{B}$ pp-definable* from $\mathbb{A}$

"0" $\mathbb{B}$ can be obtained from $\mathbb{A}$ by means of He and pp-interpretations
$\Leftrightarrow \mathbb{B} \in He(\text{pp-power}(\mathbb{A}))$

$\mathbb{A}$ <u>pp-constructs</u> $\mathbb{B}$

h1-homomorphism

a.k.a. <u>minion homomorphisms</u>

= arity preserving mapping $\xi : \mathcal{A} \to \mathcal{B}$ such that

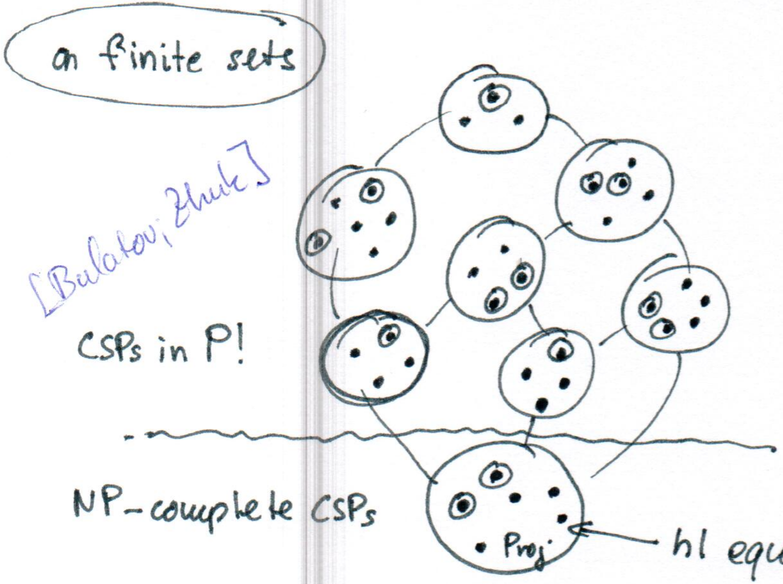
$$\xi(f(\pi^n_{i_1}, \dots, \pi^n_{i_k})) = \xi(f)(\pi^n_{i_1}, \dots, \pi^n_{i_k})$$

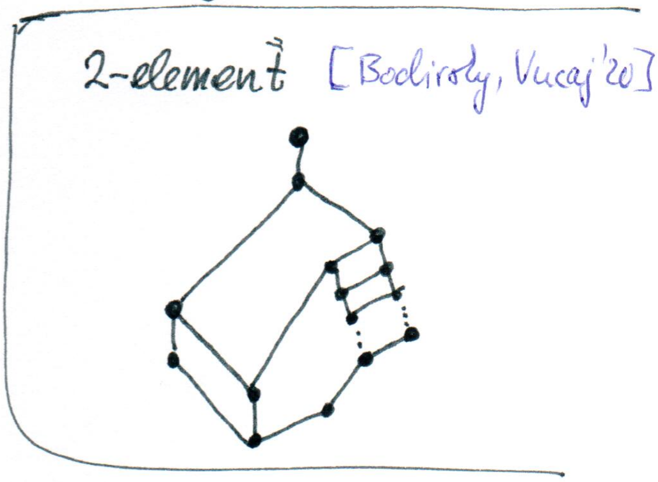
= preserves height-one identities

$$f(\text{variables}) \approx g(\text{variables})$$

When are 2 algebras essentially equal, cntd

- $\mathrm{Clo}(\underline{A}) \underset{hl}{\overset{hl}{\rightleftarrows}} \mathrm{Clo}(\underline{B})$ ⇝ ordering by $\exists$ hl homo





idempotency

⊙ idempotent

Inv $\mathscr{D}$ contains $\forall d$ $\{d\}$
- can use parameters in pp-definitions

!!! for each $\mathscr{C}$ on finite A
there exists $\mathscr{D}$ on $A' \subseteq A$ such that
- $\mathscr{D}$ is idempotent (trivial unary part!)
- $\mathscr{C} \underset{hl}{\overset{hl}{\rightleftarrows}} \mathscr{D}$

- $\mathscr{C}$ is Taylor if it is idempotent and hl equationally nontrivial
  $\Longleftrightarrow$ equationally nontrivial [Taylor 70s]
  $\Longleftrightarrow$ $\mathrm{Proj} \notin \mathrm{HSP}(\mathscr{C})$

- some tools only work for finite Taylor clones
  ☺ ok for finite domain CSPs
  ☹ bad for infinite domain CSPs

SUMMARY OF BASICS FOR FINITE

- Clones $\underset{\text{Pol}}{\overset{\text{Inv}}{\rightleftarrows}}$ Relational clones
  - often helps to use both sides (will see examples)
  - idempotency helps (can use parameters)
  - $\exists$nice operations (Mal'tsev, majority) $\leftrightarrow$ $\forall$relations are nice

- $\mathcal{A} = \mathrm{Pol}(\mathbb{A})$ $\qquad$ $\mathcal{B} = \mathrm{Pol}(\mathbb{B})$

1. $\mathbb{A}$ pp-defines $\mathbb{B}$ $\iff$ $\mathcal{A} \xrightarrow{\subseteq} \mathcal{B}$ $\iff$ $\mathcal{B} \in E(\mathcal{A})$
2. $\mathbb{A}$ pp-interprets $\mathbb{B}$ $\iff$ $\exists\, \mathcal{A} \xrightarrow{\text{homo}} \mathcal{B}$ $\iff$ $\mathcal{B} \in EHSP(\mathcal{A})$
3. $\mathbb{A}$ pp-constructs $\mathbb{B}$ $\iff$ $\exists\, \mathcal{A} \xrightarrow{\text{h1 homo}} \mathcal{B}$ $\iff$ $\mathcal{B} \in ERP(\mathcal{A})$

$\rightsquigarrow$ 3 orderings of clones

bottom:
1. clone of projections
2. equationally trivial clones
3. h1 equationally trivial clones

- Taylor = idempotent + (h1) equationally nontrivial

V COMMUTATOR THEORY

one fact will be discussed:

" $\exists$ certain invariant relation $\xRightarrow{\text{finite Taylor}}$ the clone is essentially a module "

- Clone $\mathcal{A}$ is <u>affine</u> if $Clo(\mathcal{A} + \text{constants}) = Clo(\underline{M} + \text{constants})$
  where $\underline{M}$ is an $\underline{R}$-module for some ring $\underline{R}$

  i.e. $Clo_n(\mathcal{A} + \text{constants}) = \{ r_1 x_1 + \dots r_n x_n + s ; \ r_i \in R, \ s \in M \}$

  e.g. $\mathcal{A} = Clo(\{0,1\}; \text{minor})$

- Clone $\mathcal{A}$ is <u>abelian</u> if $\forall f \in \mathcal{A} \quad a, b \in A \quad \bar{c}, \bar{d} \in A^{\cdots}$
  $$f(a, \bar{c}) = f(a, \bar{d}) \Rightarrow f(b, \bar{c}) = f(b, \bar{d})$$

  - $Clo(\underline{M}(+\text{constants}))$ for a module $\underline{M}$ is abelian
  - $\underline{G}$ group (!) $\quad Clo(\underline{G})$ abelian iff $\underline{G}$ commutative
  - $\underline{R}$ ring $\quad Clo(\underline{R})$ abelian iff $\underline{R}$ has zero $\cdot$

Relationally

- $\mathcal{A}$ abelian $\iff$ $\mathcal{A}^2$ has a congruence whose one block is $\Delta = \{(a,a) ; a \in A\}$
  - Example: $\mathcal{A} = \mathrm{Clo}(\text{module})$
    $(a_1, a_2) \sim (b_1, b_2)$ iff $a_1 - a_2 = b_1 - b_2$
  - $\iff \exists R \in \mathrm{Inv}_4(\mathcal{A})$ such that ......
  - sufficient e.g. $\exists S \in \mathrm{Inv}_3(\mathcal{A})$ "very functional", i.e.
    $\forall \{i,j,k\} = \{1,2,3\} \ \forall a_i, a_j \in A \ \exists! a_k \in A \quad (a_1 a_2 a_3) \in S$
    (Proof: $\exists u, v, u', v' \ S(u,v,x_1) \wedge S(u,v',x_2) \wedge S(u',v,y_1) \wedge S(u',v',y_2)$
    + transitive closure)

Fundamental theorem on abelian algebras

[Smith 70s]

$\mathcal{A}$ clone. $\Downarrow$

(i) $\mathcal{A}$ affine

(ii) $\mathcal{A}$ abelian and $\mathcal{A}$ has Mal'tsev operation

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Proof: (i) $\Rightarrow$ (ii) $\quad m(x,y,z) := x - y + z$ (cheating a bit)

(ii) $\Rightarrow$ (i)
- pick $0 \in A$
- $x + y := m(x, 0, y)$
- $-x := m(0, x, 0)$

$\vdots$

use the definition of Abelianess

Theorem [Hobby, McKenzie 80s]

$\mathcal{A}$ finite abelian Taylor clone $\Rightarrow$ $\mathcal{A}$ affine

Proof: use Fundamental Theorem + TCT or absorption

So, indeed, if $\mathcal{A}$ has a certain invariant relation (e.g. a "very functional" ternary relation) and is finite Taylor then $\mathcal{A}$ is essentially a module

- more generally, one defines when "congruence α centralizes β modulo γ" and define centralizers, annihilators, commutators, solvability, nilpotency(-ies) ....

  ...

- even more generally, there are higher commutators

  ....

VI TAME CONGRUENCE THEORY (TCT)

- Take finite clone $\mathcal{A}$
- Assume $\mathcal{A}$ is <u>simple</u> = no nontrivial congruences
  - this is for simplicity of presentation
- Assume $\mathcal{A}$ contains all constant operations
  - this is needed for the theory
  - → ☹ directly gives information only about reflexive relations

- $\mathcal{A}$ <u>minimal</u> if $\forall f \in \mathcal{A}_1$ is a constant or a permutation

  can be characterized! [Pálfy 80s]

assume $(|A| \geq 2)$

type 1 Clo(primitive permutation group (+ consts))

type 2 Clo(1-dim. vector space (+ consts))

type 3 $\mathrm{Clo}(\{0,1\}; \wedge, \vee, \neg, 0, 1)$ = all op.

type 4 $\mathrm{Clo}(\{0,1\}; \wedge, \vee, 0, 1)$ = monotone

type 5 $\mathrm{Clo}(\{0,1\}; \wedge, 0, 1)$, $\mathrm{Clo}(\{0,1\}, \vee, 0, 1)$

- if $\mathcal{A}$ is not minimal

  - define $\mathcal{M}$ = minimal members of $\{f(A);\ f \in \mathcal{A}_1,\ |f(A)| \geq 2\}$

  - for $M \in \mathcal{M}$ $\quad \mathcal{A}{\restriction}M = \{f{\restriction}_{M^n};\ f \in \mathrm{Clo}_n \mathcal{A} \text{ preserves } M\}$

    --- clone on M $\quad (\mathcal{A} \xrightarrow{\text{nl homo}} \mathcal{A}{\restriction}M)$

    it is minimal $\longrightarrow$ type 1–5

  - (neigborhood) $\forall M \in \mathcal{M}\ \exists e \in \mathcal{A}_1\ \ e(A) = M\ \ e{\restriction}_M = id$

    $\rightsquigarrow$ nicely working relativitization of pp-formulas

    e.g. $\exists y \in A\ R(x,y) \wedge S(y,z)$ on M

    $= \exists y \in M\ R(x,y) \wedge S(y,z)$ on M

  - (isomophism) $\forall M, N \in \mathcal{M}\ \ \mathcal{A}{\restriction}M \doteq \mathcal{A}{\restriction}N$

    up to renaming provided by $\mathcal{A}_1$

    $\rightsquigarrow$ $\mathcal{A}$ has type

  - (separation) $\forall M \in \mathcal{M}\ \forall a \neq b \in A\ \exists f \in \mathcal{A}_1$
    $f(A) \subseteq M$ and $f(a) \neq f(b)$

    $\rightsquigarrow$ $\mathcal{A}$ "embeds" into a power of M

  - (connectivity) $\forall a, b \in A$ connected by members of $\mathcal{M}$

    $\rightsquigarrow$ helps in local $\rightarrow$ global

[Hobby, McKenzie 80s]

- if $\mathcal{A}$ is not simple, types relative to "tame intervals"

VII BULATOV'S THEORY

instead of minimal sets one looks at 2-generated sets

- $\mathcal{A}$ finite clone on A $\rightsquigarrow$ colored digraph on A

$a \xrightarrow{\text{semilattice}} b$ if $\exists \theta$ proper congruence of $Sg(a,b)$

$\exists f \in \mathcal{A}_2$

$(\{a/\theta, b/\theta\}, f)$ makes sense and

$\doteq (\{0,1\}, \vee)$

$a \overset{\text{majority}}{\longleftrightarrow} b$ —— " ——

$\exists f \in \mathcal{A}_3$

$(\{a/\theta, b/\theta\}, f)$ —" —

$\doteq (\{0,1\}, maj)$

$a \overset{\text{affine}}{\longleftrightarrow} b$ - - - - - - - -

$(G; x-y+z)$ $\underline{G}$ abelian group

- $\rightsquigarrow$ locally nice properties of relations

- (connectivity) $\mathcal{A}$ Taylor $\Rightarrow$ the digraph is connected

VIII ABSORPTION THEORY

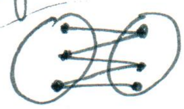
~ideals for general algebras

somewhat nicely working relativization of pp-formulas

- $\mathcal{A}$ idempotent clone
- $B \overset{\text{absorbs}}{\trianglelefteq} \mathcal{A}$ if $\underset{\neq\emptyset}{B} \leq \mathcal{A}$ and $\exists t \in \mathcal{A}$
  $t(B, \dots, B, A, B, \dots, B) \subseteq B$
- not rare: [B, Kozik]

$\boxed{\text{Theorem}}$: $\mathcal{A}$ finite Taylor, $R \in \mathrm{Inv}_2 \mathcal{A}$ linked
$\Rightarrow \exists B \underset{\neq}{\trianglelefteq} \mathcal{A}$

Proof: combination of composition + pp-definitions



- consequences (with some work)
  - $\mathcal{A}$ finite Taylor, $R \in \mathrm{Inv}_2 \mathcal{A}$ linked $\Rightarrow$ R has a loop
  - [Kearnes, Marković, McKenzie] $\mathcal{A}$ finite idempotent. $\mathcal{A}$ Taylor
    $\Leftrightarrow \exists s \in \mathcal{A}_4 \quad s(r,a,r,e) \approx s(a,r,e,a)$
  - [Zhuk] $\mathcal{A}$ finite simple Taylor
    - $\leftarrow$ no irredundant subdirect invariant relations
    - $\mathcal{A}$ affine
    - $\exists B \underset{\neq}{\trianglelefteq} \mathcal{A}$

(also • $\mathcal{A}$ finite Taylor abelian $\Rightarrow$ $\mathcal{A}$ Mal'tsev ($\Rightarrow$ affine))

TODOs

- organize the mess, some progress
  - Ross Willard's work
  - minimal Taylor clones
    [LB, Brady, Bulatov, Kozik, Zhuk]
- generalize
  - oligomorphic clones = Pol (ω-categorical)
  - weighted relations
  - minions (set ~~of~~ operations $A^n \to B$)
- more abstract nonsence – e.g.
  weaker orderings of clones