Decidability of absorption for relational structures

Libor Barto joint work with Jakub Bulín

Charles University in Prague

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Two problems

Problem (NU problem for algebras)

Given a finite algebra $\mathbf{A}$ *is it decidable whether* $\mathbf{A}$ *has an NU operation?*

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Problem (NU problem for relational structures)

Given finite relational structure $\mathbb{A}$ *is it decidable whether* $\mathrm{Pol}(\mathbb{A})$ *has an NU operation?*

- In EXPTIME Barto
- **Why?** Because for finitely related algebras **A** has NU iff **A** has Jónsson operations (there is a generalization...)

Harder (and "better") problems

Problem (Absorption for algebras)

Given a finite algebra $\mathbf{A}$ *and* $B \subseteq \mathbf{A}$ *is it decidable whether* B *absorbs* $\mathbf{A}$*?*

- It is decidable if $|B| = 1$ (Horowitz, Valeriote)
- General case still open (likely in EXPTIME)

Problem (Absorption for relational structures)

Given finite relational structure $\mathbb{A}$ *and* $B \subseteq A$ *is it decidable whether* B *absorbs* $\mathrm{Pol}(\mathbb{A})$*?*

- It is decidable Bulín if $\mathrm{Pol}(\mathbb{A})$ is $SD(\wedge)$
- In EXPTIME Barto, Bulín
- **Why?** Because of the result in this talk

Absorption (a generalization of NU)

Definition

B absorbs $\mathbf{A}$, written $B \triangleleft \mathbf{A}$, if $\exists$ idempotent term t such that $t(B, B, \ldots, B, A, B, \ldots, B) \subseteq B$.

Example: $\mathbf{A}$ has an NU iff every singleton absorbs $\mathbf{A}$.

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The problem of deciding absorption also came up naturally

directed Jónsson terms

Theorem (Kozik)

For finite $\mathbf{A}$, $HSP(\mathbf{A})$ *is congruence distributive iff there are idempotent terms such that*

$x \approx p_0(x, y, z), z \approx p_n(x, y, z)$

$p_i(x, y, y) \approx p_{i+1}(x, x, y)$

$p_i(x, y, x) \approx x$

directed Jónsson ideals

Definition

For finite **A** and $B \leq \mathbf{A}$, B is a Jónsson ideal of **A**, written $B \triangleleft_j \mathbf{A}$ iff there are idempotent terms such that

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Fact: $HSP(\mathbf{A})$ is CD iff every singleton is a Jónsson ideal of **A**

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Fact: $B \triangleleft \mathbf{A} \;\Rightarrow\; B \triangleleft_j \mathbf{A}$

Fact: The other implication fails, BUT

The result

Recall: $\mathbf{A}$ is finitely related if $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$ for $\mathbb{A}$ with finitely many relations

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Proof uses techniques from "$CD \Rightarrow NU$" and a paper by Zhuk

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The relational version known to be in P for

- posets Kun, Szabó 01
- reflexive graphs Larose, Loten, Zádori 05
- reflexive digraphs Maróti, Zádori 12

Relational characterization of absorption

$\mathbf{A}$ . . . finite idempotent algebra

Theorem

$\mathbf{A}$ has NU of arity n iff every $R \leq \mathbf{A}^n$ is determined by projections to $n-1$ coordinates.

A similar characterization

Theorem

$B \triangleleft \mathbf{A}$ wrt. term of arity n iff there is no relation $R \leq \mathbf{A}^n$ such that

- *R does not intersect B^n*
- *each projection of R to $n-1$ coordinates intersect B^{n-1}*

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Thank you!