# A Hardness Criterion for Promise CSPs

Libor Barto
joint work with J. Bulín, A. Krokhin, and J. Opršal

Department of Algebra, Charles University, Prague

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**CSPs over fixed finite templates**

- class of computational problems
- tight link to universal algebra, namely “special” Maltsev conditions [Jeavons’98], [Bulatov, Jeavons, Krokhin’05], [Barto, Pinsker’18]
- simple criterion for hardness [Bulatov, Jeavons, Krokhin’05]
- good enough [Bulatov’17], [Zhuk’17]

**PCSPs over fixed finite templates**

- larger class of computational problems
- discovered tighter link to universal algebra *
- the same simple criterion for hardness...
- ...but it is not good enough, stronger exists *

* this talk

Fix $\mathbb{A}$: finite relational structure

**Definition ($\mathrm{CSP}(\mathbb{A})$)**

**Input:** $\mathbb{X}$ of the same signature as $\mathbb{A}$
**Yes:** there is a homomorphism $\mathbb{X} \to \mathbb{A}$
**No:** there is none

**Example:** $\mathrm{CSP}(\mathbb{K}_3) =$ decide whether a given graph is 3-colorable

The computational complexity of $\mathrm{CSP}(\mathbb{A})$ depends only on

- $\mathcal{M} = \mathrm{Pol}(\mathbb{A})$, the polymorphism clone [J'98]
- strong Maltsev conditions satisfied by $\mathcal{M}$ [BJK'05]
- minor conditions satisfied by $\mathcal{M}$ [BP'18] [BBKO]

Minor condition: A set of identities of the from

$$symbol(variables) \approx symbol(variables)$$

**Example:** $f(x,x,y) \approx g(y,x)$, $g(x,y) \approx h(x,y,y,x)$,
$g(x,y) \approx g(y,x)$, ...

Minor condition is satisfied by $\mathcal{M}$
(where $\mathcal{M}$ is a set of operations on a fixed set)
if there exist $f, g, h$ in $\mathcal{M}$ making the identities true
(a solution in $\mathcal{M}$)

It is trivial if it is satisfied by every clone (=by projections)

**Corollary:** If Pol($\mathbb{A}$) satisfied only trivial minor conditions,
then $\mathrm{CSP}(\mathbb{A})$ is NP-complete

**Theorem [B'17,Z'17]:** Otherwise $\mathrm{CSP}(\mathbb{A})$ is in P

**Definition** ($\mathrm{MinorCond}(N, \mathcal{M})$)

**Input:** minor condition with symbols of arity $\leq N$
**Yes:** it is trivial
**No:** it is not satisfied by $\mathcal{M}$

**Theorem** ([BBKO])

*Let $\mathcal{M} = \mathrm{Pol}(\mathbb{A})$. The following computational problems are equivalent for a large enough $N$.*

(i) $\mathrm{CSP}(\mathbb{A})$
(ii) $\mathrm{MinorCond}(N, \mathcal{M})$

**Consequence:** The computational complexity of $\mathrm{CSP}(\mathbb{A})$ depends only on minor conditions satisfied by $\mathcal{M}$
**Proof:** direct, simple, known

Fix $\mathbb{A}$, $\mathbb{B}$: finite relational structures with $\mathbb{A} \to \mathbb{B}$

**Definition** ($\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$)

**Input:** $\mathbb{X}$ of the same signature as $\mathbb{A}$
**Yes:** there is a homomorphism $\mathbb{X} \to \mathbb{A}$
**No:** there is no homomorphism $\mathbb{X} \to \mathbb{B}$

**Example:** $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_7) =$ distinguishing between 3-colorable and
not 7-colorable graphs.

Search version: find a 7-coloring of a given 3-colorable graph

$\mathcal{M} = \mathsf{Pol}(\mathbb{A}, \mathbb{B})$ is the polymorphism minion
functions that map tuples in $R^{\mathbb{A}}$ to tuples in $R^{\mathbb{B}}$

**Theorem:** $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is equivalent to $\mathrm{MinorCond}(N, \mathcal{M})$
for a large enough $N$

# A criterion for hardness



## Corollary

*If* Pol$(\mathbb{A}, \mathbb{B})$ *satisfies only trivial minor conditions, then* $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ *is NP-complete.*

## Theorem (folklore)

*TFAE for a minion* $\mathcal{M}$.

- $\mathcal{M}$ *satisfies only trivial minor conditions*
- *There is a mapping* $\xi : \mathcal{M} \to \mathbb{N}$ *such that*
  - *if* $f$ *is of arity* $n$*, then* $\xi(f) \in \{1, 2, \dots, n\}$ *(**think**: an important coordinate of* $f$*)*
  - $\xi$ *behaves nicely with minors, eg. if*

$$f(x_3, x_2, x_1, x_2, x_2, x_1) = g(x_1, x_2, x_3)$$

*and* $\xi(f) = 5$*, then* $\xi(g) = 2$.

# A criterion for hardness, cntd'

**Theorem (repeated)**

*If there is a mapping* $\xi : \mathrm{Pol}(\mathbb{A}, \mathbb{B}) \to \mathbb{N}$

- *if f is of arity n, then* $\xi(f) \in \{1, 2, \ldots, n\}$
  *(**think**: an important coordinate of f)*
- $\xi$ *behaves nicely with minors*

*then* $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ *is NP-complete.*

- for $\mathbb{A} = \mathbb{B}$, ie. $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ is a clone,
  there are many equivalent characterizations
  (TCT type 1, no Taylor, no Siggers, no weak NU, no cyclic...)
- for $\mathbb{A} = \mathbb{B}$ the criterion is good enough
- good enough to prove $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_4)$ NP-complete
  [Brakensiek, Guruswami'16]
- not good enough to prove $\mathrm{PCSP}(\mathbb{C}_{137}, \mathbb{K}_3)$ NP-complete
- not good enough to prove $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$ NP-complete

**Definition** ($\mathrm{MinorCond}(N, \varepsilon)$)

**Input:** minor condition with symbols of arity $\leq N$
**Yes:** it is trivial
**No:** no $\varepsilon$–fraction of identities is trivial

Two famous theorems in computational complexity

- PCP theorem [Arora, Lund, Motwani, Sudan, Szegedy'98]
- Parallel repetition theorem [Raz'98]

give the following theorem.

**Theorem**

*For each $\varepsilon > 0$ there exists $N$ such that $\mathrm{MinorCond}(N, \varepsilon)$ is NP-complete.*

## Corollary

*If there exists* $C \in \mathbb{N}$ *and a mapping* $\xi : \mathrm{Pol}(\mathbb{A}, \mathbb{B}) \to P(\mathbb{N})$ *such that*

- *if* $f$ *is of arity* $n$*, then* $\xi(f) \subseteq \{1, 2, \ldots, n\}$, $|\xi(f)| \leq C$ *(**think**: a small set of important coordinates of* $f$*)*
- $\xi$ *behaves nicely with minors, eg. if*

$$f(x_3, x_2, x_1, x_2, x_2, x_1) = g(x_1, x_2, x_3)$$

*and* $\xi(f) = \{4, 5, 6\}$*, then* $\xi(g) \cap \{1, 2\} \neq \emptyset$

*Then* $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ *is NP-complete.*

# Proof of the hardness criterion



- we have $\xi : \mathcal{M} \to P(\mathbb{N})$, want to show that
  (a) $\mathrm{MinorCond}(N, 1/C^2)$ reduces to
  (b) $\mathrm{MinorCond}(N, \mathcal{M})$ (via trivial reduction)
- **Recall:**
  - **Input:** bipartite minor condition (symbols of arity $\leq N$)
  - **Answer Yes:** it is trivial
  - **Answer No:**
    (a) no $1/C^2$–fraction of identities is trivial
    (b) not satisfied in $\mathcal{M}$
- "Yes input $\to$ Yes input": trivial
- "No input $\to$ No input": for contrapositive:
  - take a valid interpretation in $\mathcal{M}$
  - reinterpret $f$ as the $i$-th projection, where $i \in \xi(f)$ random
  - each equation is satisfied with probability $\geq 1/C^2$
  - so expected fraction of satisfied equations is $\geq 1/C^2$
  - so some $1/C^2$-fraction is trivial

**Corollary (repeated)**

*If there exists $C \in \mathbb{N}$ and a mapping $\xi : \mathrm{Pol}(\mathbb{A}, \mathbb{B}) \to P(\mathbb{N})$ such that*

- *if $f$ is of arity $n$, then $\xi(f) \subseteq \{1, 2, \ldots, n\}$, $|\xi(f)| \leq C$ (**think**: a small set of important coordinates of $f$)*
- *$\xi$ behaves nicely with minors*

*Then $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is NP-complete.*

- good enough for all known NP-complete PCSPs over 2-element domains
- good enough to prove $\mathrm{PCSP}(\mathbb{C}_{137}, \mathbb{K}_3)$ NP-complete [Krokhin, Opršal'19]
- not good enough to prove $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$ NP-complete

**Theorem** ([BBKO])

*If* $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_n$ *and for each* $i$ *there exists* $C : \mathbb{N} \to \mathbb{N}$ *and a mapping* $\xi : \mathcal{M}_i \to P(\mathbb{N})$ *such that*

- *if* $f$ *is of arity* $n$, *then* $\xi(f) \subseteq \{1, 2, \ldots, n\}$, $|\xi(f)| \leq C(n)$
- $C(n) = o(n^\alpha)$ *for each* $\alpha > 0$ *(eg.* $C(n) \leq 100 \log^5(n)$*)*
- $\xi$ *behaves nicely with minors, eg. if*

$$f(x_3, x_2, x_1, x_2, x_2, x_1) = g(x_1, x_2, x_3)$$

*where* $f, g \in \mathcal{M}_i$ *and* $\xi(f) = \{4, 5, 6\}$, *then* $\xi(g) \cap \{1, 2\} \neq \emptyset$

*Then* $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ *is NP-complete.*

**Proof:** Using NP-hardness of "Layered Gap Label Cover"

[Dinur, Guruswami, Khot, Regev'05]

# The criterion: applications, comments



- good enough to prove NP-completeness of $\mathrm{PCSP}((\{0,1\};\{\overset{\frown}{001}\}),(\{0,1,2\};\{\overset{\frown}{001},\overset{\frown}{002},\overset{\frown}{112}\})$ [Diego's talk]
- good enough to prove $\mathrm{PCSP}(\mathbb{K}_3,\mathbb{K}_5)$ NP-complete [BKO]
- **Q:** Is it good enough to prove NP-completeness of every NP-complete PCSP???
- **Q:** Is there a nicer criterion?
- for $\mathbb{A}=\mathbb{B}$, ie. Pol$(\mathbb{A},\mathbb{B})$ is a clone, the condition is equivalent to satisfying only trivial minor conditions.

  this follows from

  - $P \neq NP$ and the CSP dichotomy theorem or
  - a result about cyclic operations [Barto, Kozik'12]
  - **Q:** $\exists$ more elementary proof?

- good enough to prove NP-completeness of $\mathrm{PCSP}((\{0,1\};\{\overset{\curvearrowleft}{001}\}),(\{0,1,2\};\{\overset{\curvearrowleft}{001},\overset{\curvearrowleft}{002},\overset{\curvearrowleft}{112}\})$ [Diego's talk]
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  this follows from
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  - a result about cyclic operations [Barto, Kozik'12]
  - **Q:** $\exists$ more elementary proof?

**Thank you for your patience!**