CSPs & Set Functors

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Structure Meets Power
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OUTLINE

1. THE GOAL AND THE STRATEGY

2. simple categorial < complexity invariant / formulation

   of PCSP(finite, finite)

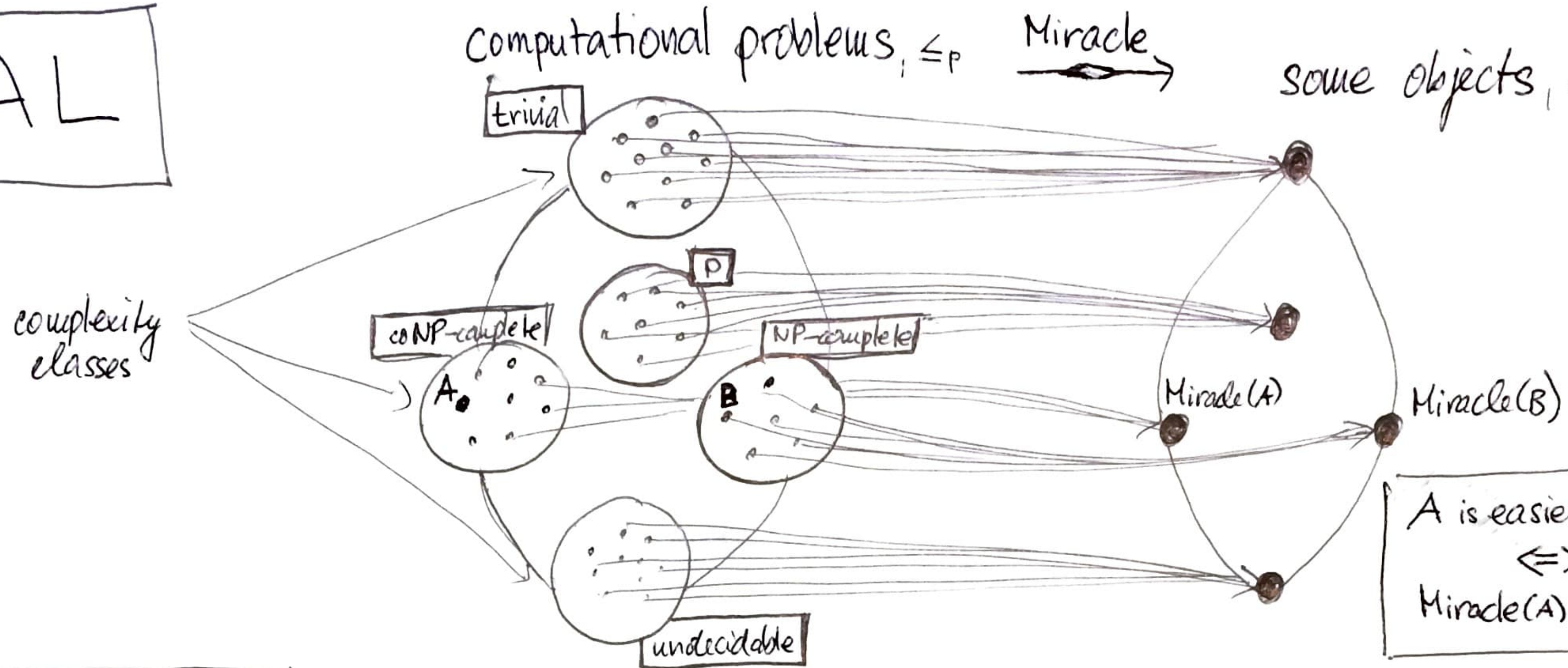
3. "finite" formulation of CSP(ω-categorical)

THE GOAL

&

THE STRATEGY

GOAL



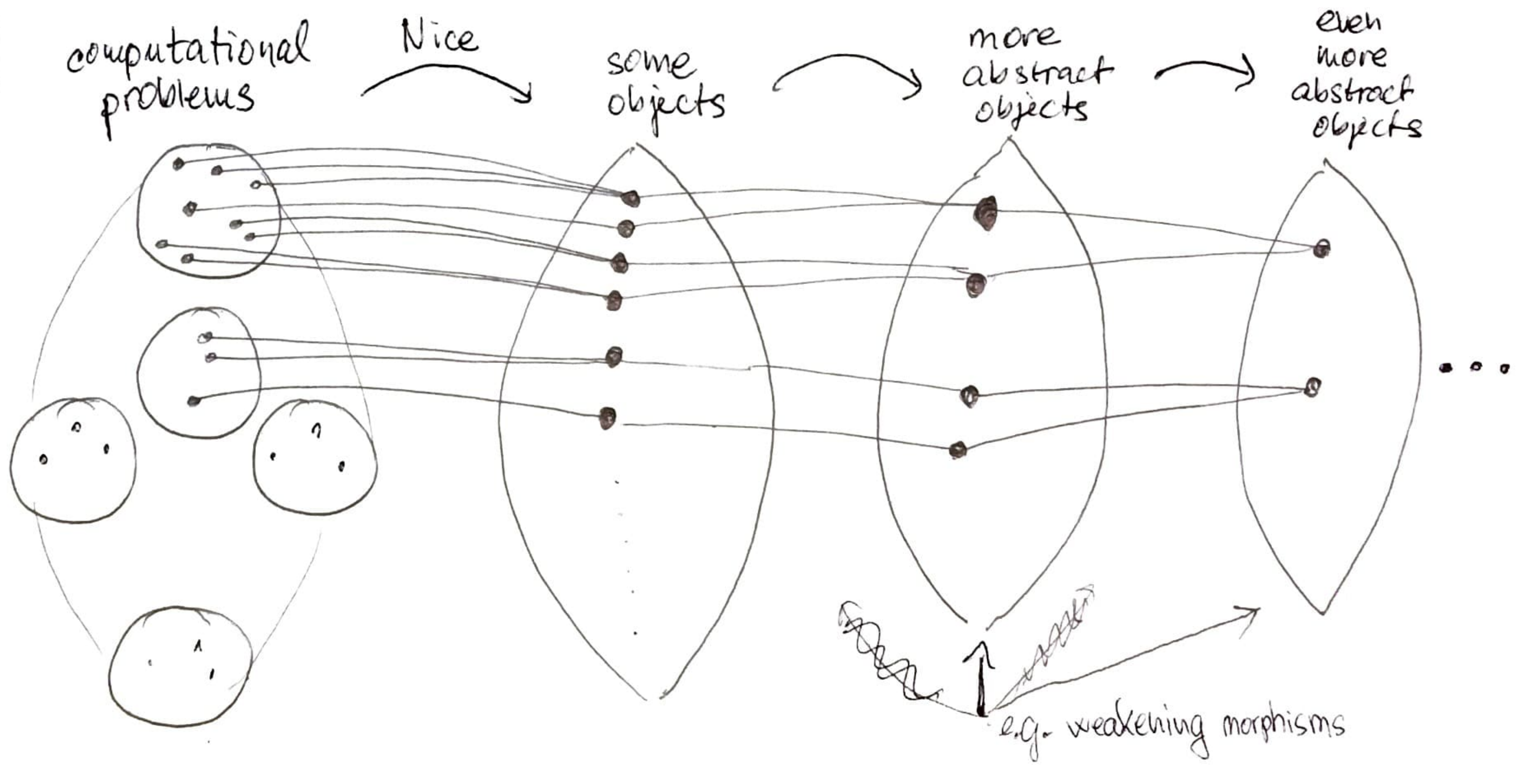
goal

A is easier than B

$\Longleftrightarrow$

Miracle(A) $\geq$ Miracle(B)

STRATEGY



want

A is easier than B

$\Uparrow$

image(A) $\geq$ image(B)

PCSP(finite, finite)

CSP

$\mathbb{A} = (A;\ R \subseteq A^3,\ S \subseteq A^2, \dots)$ rel. str.

(now finite)

CSP($\mathbb{A}$)

INPUT: finite $\mathbb{X}$ similar to $\mathbb{A}$

YES: $\mathbb{X} \to \mathbb{A}$

NO: $\neg\, \mathbb{X} \to \mathbb{A}$

Examples for finite $\mathbb{A}$

- 3-SAT, ...
- 3-COLORING
- $\mathbb{Z}_2$-LINEAR EQUATIONS

Good starting point for THE GOAL

PCSP

$\mathbb{A} = (A;\ R \subseteq A^3,\ S \subseteq A^2, \dots)$

$\downarrow$ finite

$\mathbb{B} = (B;\ R' \subseteq B^3,\ S' \subseteq B^2, \dots)$

PCSP($\mathbb{A}, \mathbb{B}$)

INPUT: finite $\mathbb{X}$ similar to $\mathbb{A}, \mathbb{B}$

YES: $\mathbb{X} \to \mathbb{A}$

NO: $\neg\, \mathbb{X} \to \mathbb{B}$

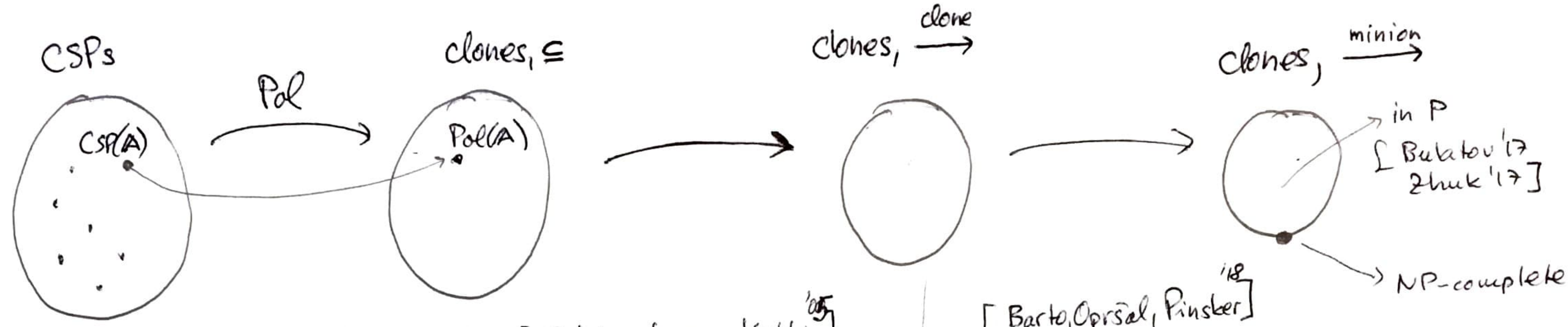
Example for finite $\mathbb{A}, \mathbb{B}$

- 3 vs. 6 - COLORING

Great opportunity for THE GOAL

+ includes <u>concrete problems of major interest</u> in CS

CSP & THE STRATEGY



[Jeavons et al 98]

[Bulatov, Jeavons, Krokhin '05]

[Barto, Opršal, Pinsker '18]

$$\mathrm{Pol}(\mathbb{A}) = \{ f : A^n \to A \,;\; n \in \mathbb{N},\; f : \mathbb{A}^n \to \mathbb{A} \}$$

<u>clone</u> on A

$$\mathcal{C} \longrightarrow \mathcal{D}$$
$$f \longmapsto \bar{f}$$

is a <u>clone homo</u> if it preserves term definitions

e.g. $f(x,y) = g(h(x), y, x)$ in $\mathcal{C}$

$\Rightarrow \bar{f}(x,y) = \bar{g}(\bar{h}(x), y, x)$ in $\mathcal{D}$

$$\mathcal{C} \longrightarrow \mathcal{D}$$
$$f \longmapsto \bar{f}$$

is a <u>minion homo</u> if it preserves minors

e.g. $f(x,y) = g(x,y,x)$ in $\mathcal{C}$

$\Longrightarrow \bar{f}(x,y) = \bar{g}(x,y,x)$

see also

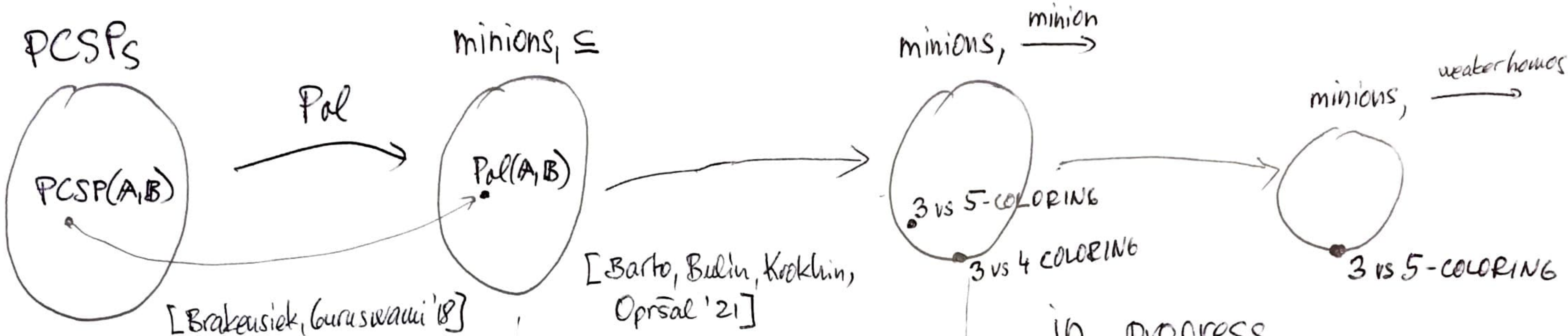
abstract clones
monads
varieties

functors Set → Set !

☺ dichotomy ☹ not yet

triv
P-complete
NP-complete

PCSP & THE STRATEGY



in progress...
[Barto, Kozik '22]
[Opršal's talk]

$$\mathrm{Pol}(\mathbb{A},\mathbb{B}) = \{ f : A^n \to B ;\ n \in \mathbb{N},\ f : \mathbb{A}^n \to \mathbb{B} \}$$

<u>minion</u> on (A, B)

= closed under taking minors

e.g. $g \in \mathcal{M} \Rightarrow f \in \mathcal{M}$ where

$f(x,y) := g(x,y,x)$

<u>minion homo</u>

= preserves minors
(as before)

functors Set → Set
+ natural transformations

!

☹ not enough even for NP-completeness

☺ great for the GOAL

PCSPs $\longrightarrow$ SET functors

$$\mathrm{PCSP}(\mathbb{A},\mathbb{B}) \longmapsto \overset{\mathrm{Pol}^F(\mathbb{A},\mathbb{B})}{\cancel{\mathrm{Pol}}} : \mathcal{S}et \longrightarrow \mathcal{S}et$$

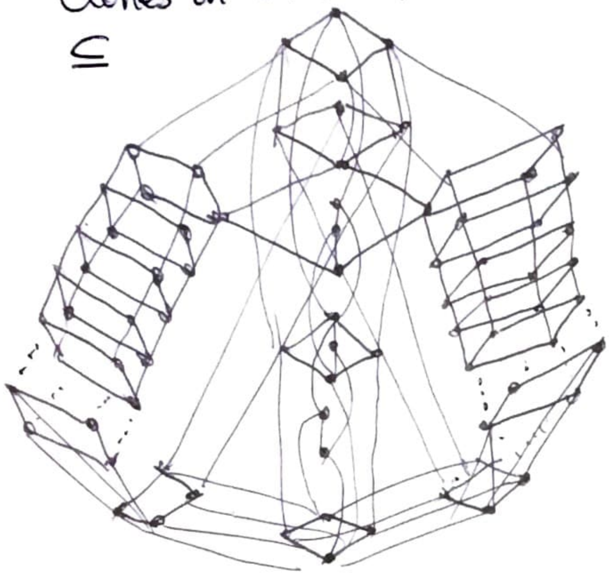
$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "\alpha"] & \{\mathbb{A}^X \longrightarrow \mathbb{B}\} \arrow[d, "\text{taking minors (as } B^{A^-})"] \\ Y \arrow[r, mapsto] & \{\mathbb{A}^Y \longrightarrow \mathbb{B}\} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Pol}^F(\mathbb{A},\mathbb{B}) \arrow[d, "\text{natural transformation}"] & \Longrightarrow & \mathrm{CSP}(\mathbb{A},\mathbb{B}) \\ \mathrm{Pol}^F(\mathbb{C},\mathbb{D}) & & \leq_{\text{log-space}} \mathrm{CSP}(\mathbb{C},\mathbb{D}) \end{tikzcd}$$

PICTURES

Clones on $\{0,1\}$ [Post 40's]

$\subseteq$



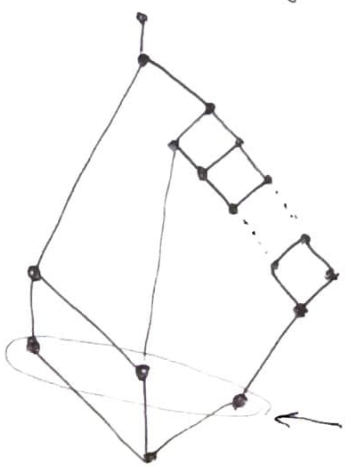
clones on $\{0,1,2\}$, $\subseteq$

clones on $\{0,1,2\}$ determined by binary relations, $\subseteq$

$\sim$ 2M clones

Clones on $\{0,1\}$ [Bodirsky, Vucaj '20]

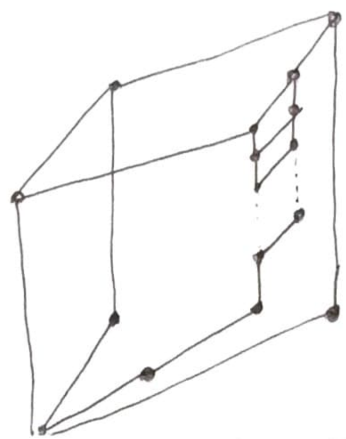
$\xrightarrow{\text{minor}}$



HORN-SAT
2-SAT
$\mathbb{Z}_2$-LINEAR-EQ

NP-complete

The same interval, $\xrightarrow{\text{minor}}$



[Bodirsky, Vucaj, Zhuk]

dtto, $\xrightarrow{\text{minor}}$

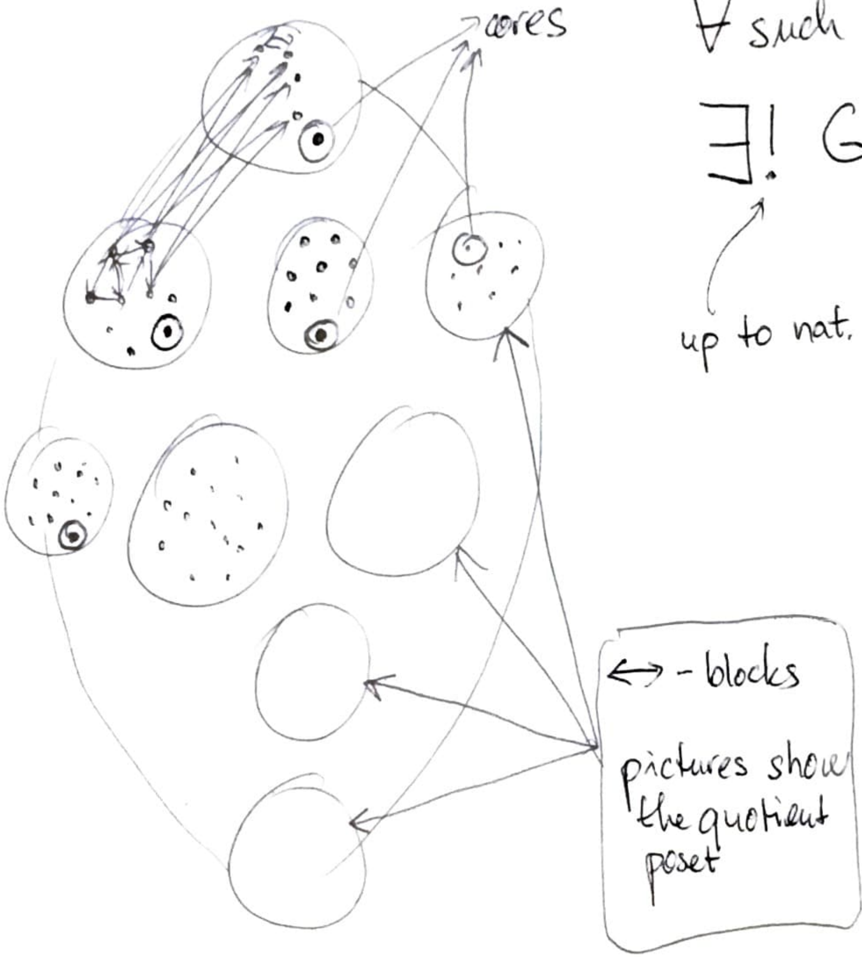
< 250

in progress...

NOTE ON CORES

Functors $\mathrm{Pol}^F(\text{finite}, \text{finite})$

$\xrightarrow{\text{natural transformation}}$



$\forall$ such $F\colon \mathbf{Set} \to \mathbf{Set}$

$\exists!\ G\colon \mathbf{Set} \to \mathbf{Set}$ such that

up to nat. iso.

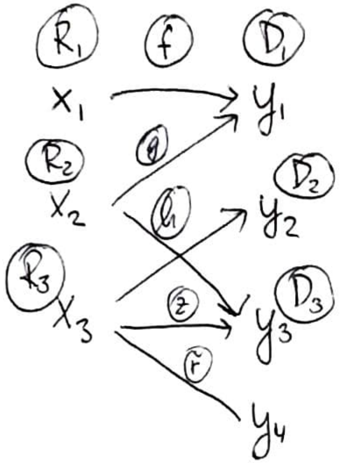
- $F \rightleftarrows G$
- G is a <u>core</u>, i.e. every $G \to G$ is iso.

PCSP(functor)

$F : \mathcal{Set} \to \mathcal{Set}$

PCSP(F, n)

INPUT: e.g.



$f : R_1 \to D_1$ mapping

$|R_1|, |D_1| \leq n$

etc.

YES: $\exists s(x_1) \in R_1 \quad \exists s(x_2) \in R_2 \ \dots$

$\exists s(y_1) \in D_1 \ \dots$

$f(s(x_1)) = s(y_1), \ \dots$

NO: $\neg \quad \exists s(x_1) \in FR_1 \ \dots$

$\exists s(y_1) \in FD_1 \ \dots$

$Ff(s(x_1)) = s(y_1) \ \dots$

Theorem [BBKO]

$$\mathrm{PCSP}(\mathbb{A}, \mathbb{B}) \sim_{\text{log-space}} \mathrm{PCSP}(\mathrm{Pol}^F(\mathbb{A}, \mathbb{B}), n)$$

for sufficiently large n

Examples

- $F = P$ (power set) HORN-SAT
- $F = P_{odd}$ $\mathbb{Z}_2$-LINEAR-EQUATIONS
- $F \subseteq P^- \circ P^-$ 2-SAT

PCSP(fin, fin) SUMMARY

- each PCSP(finite, finite) is log-space equivalent to PCSP(functor Set → Set)
- natural transformation gives reduction
- need to weaken natural transformations further

CSP(ω-categorical)

CSP(infinite)

- CSP($\mathbb{A}$) makes sense for infinite $\mathbb{A}$ (finite signature)

- $\mathbb{A}$ ω-categorical & countable $\Rightarrow$ [Bodirsky, Nešetřil '06]
  - Pol($\mathbb{A}$) is a complexity invariant but not $\mathrm{Pol}^F(\mathbb{A})$ [Gillibert, Jonušas, Kompatscher, Mottet, Pinsker '20]
  - rich complexity-wise

  $\forall n$ $\mathrm{Aut}(\mathbb{A}) \curvearrowright A^n$ has finitely many orbits

- $\mathbb{A}$ reduct of finitely bounded homogeneous structure: back in NP

  Bodirsky-Pinsker dichotomy conjecture $<$ NP-complete / P

- CSP(ω-categorical) can be phrased as a "finite" CSP

G-CSP

$G : \mathcal{S}et \to \mathcal{S}et$ finitary

G-template $\mathbb{T} = \{ (S, R \subseteq GS), \dots \}$

finite

G-CSP($\mathbb{T}$)

INPUT: • finite V

• constraints $\begin{cases} (S_1, R_1) \in \mathbb{T} \\ i_1 : S_1 \to V \end{cases}$

....

YES: $\exists x \in GV \quad Gi_1(x) \in R_1, \dots$

Examples

• $G = A^{-}$ for finite A

$G\text{-CSP}(\mathbb{T}) \doteq \mathrm{CSP}(\mathbb{A})$ naturally defined

• $\mathbb{A} = (\mathbb{Q}, \text{"x between y and z"})$

$GX = A^X / \text{orbits of } \mathrm{Aut}(\mathbb{Q}, \leq)$

$G[1] = \{ \bullet \}$

$G[2] = \{ \; , \; , \; \}$

$G[3] = \{ \; , \; , \; , \; , \dots \}$

$\mathbb{T} = \{ ([3], \{ \; , \; \}) \}$

$G\text{-CSP}(\mathbb{T}) \doteq \mathrm{CSP}(\mathbb{A})$

• A reduct of ...... : translates to property of G

? Better framework? Is it more general? Polymorphisms???

Thank you!