Infinite domain constraint satisfaction problem

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CSP

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Constraint Satisfaction Problem

CSP=CS+L

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Fixed-template CSP over **finite** domains

- **C**omputer **S**cience
  - CSP is a class of computational problems within NP
- **L**ogic
  - approached via logic, mostly universal algebra

Achievements

- A lot of tractability/hardness results under one roof
  – such a generality and strength was not dreamt of
- Better understanding why problems are hard (lack of symmetry)
- (meta) People from different fields talk
- Distant areas coming closer
- Expands: counting, optimization, approximation, **infinite domains**, ...

CSP=CS+L, infinite

Fixed-template CSP over **infinite** domains

- Covers all computational problems (too optimistic to attack now)
- But there are reasonable restrictions (eg. to be back in NP)
- Covers way more computational problems in NP than CSP over finite domains
- More areas of mathematics (mostly logicky) involved
  - universal algebra
  - model theory (ω-categorical structures, homogeneity)
  - Ramsey theory ($\leftrightarrow$ abstract topological dynamics)
- More people talk to each other, etc

Outline

- Some general propaganda (it's over)
- CSP and examples
- Complexity classification conjectures
- News

Disclaimer

- No pictures
- Most of the definitions imprecise
- Almost no theorem true as stated

CSP and examples

Fixed-template CSP

Fix $\mathbb{A} = (A; R_1, R_2, \ldots, R_n)$: relational structure (each $R_i \subset A^k$)

Definition

Instance of $\mathrm{CSP}(\mathbb{A})$: Expression of the form

$$R_1(x, y, z),\ R_2(t, z),\ R_1(y, y, z),\ \ldots$$

where each R_i is in $\mathbb{A}$.

Question: Is there a map: variables $\to A$ satisfying all constraints?

- Other variants: nothing is fixed, something else is fixed (includes industrial problems)
- Even within fixed–template, many other questions, eg.
  - Count the number of solutions
  - Optimize the number of satisfied constraints
  - Approximately optimize the number of satisfied constraints
- This talk: only the above version, only computational complexity

Finite examples

Boolean satisfiability problems, (di)graph coloring problems, solving systems of equations over finite domain

- 3-SAT: $\mathbb{A} = (\{0,1\}; x \vee y \vee z, x \vee y \vee \neg z, \dots)$
- 3-COLOR: $\mathbb{A} = (\{0,1,2\}; x \neq y)$
- q-LIN: $\mathbb{A} = (GF(q); \textit{affine subspaces})$

Infinite examples

Problems in temporal and spatial reasoning, and many more AI areas, computational linguistics, phylogenetic reconstruction (see Bodirsky'12)

- ACYCLICITY: $\mathbb{A} = (\mathbb{Q}; x < y)$
- BETWEENNESS: $\mathbb{A} = (\mathbb{Q};\ y$ between x and $z)$
- GRAPH-SAT(F), where F is a set of finite graph formulas

  INPUT: Φ_1(variables), Φ_2(variables), ... with $\Phi_i \in F$

  OUTPUT: Is it satisfiable in a graph?

Complexity classification conjectures

Relational side

- **Crucial:** If $\mathbb{A}$ pp-interprets $\mathbb{B}$, then $\mathrm{CSP}(\mathbb{B})$ is easier than $\mathrm{CSP}(\mathbb{A})$
  - $\mathbb{A}$ pp-defines $\mathbb{B}$ = the same domain, and relations in $\mathbb{B}$ definable using relations in $\mathbb{A}$, and $\exists, =, \wedge$.
  - $\mathbb{A}$ pp-interprets $\mathbb{B}$ if the domain of $\mathbb{B}$ is a pp-definable subset of A^k modulo a pp-definable equivalence and all the relations of $\mathbb{B}$ are "pp-definable" from $\mathbb{A}$
- **Corollary:** If $\mathbb{A}$ pp-interprets some structure with NP-hard CSP (like 3–SAT), then $\mathrm{CSP}(\mathbb{A})$ is NP-hard
- **Remark:** $\mathbb{A}$ pp-interprets 3–SAT, then it pp-interprets every finite structure
- **Tractability conjectures:** If $\mathbb{A}$ is (*) and does not interpret 3–SAT then $\mathrm{CSP}(\mathbb{A})$ is in P
  - (*) finite [Feder,Vardi'93],[Bulatov,Jeavons,Krokhin'00]
  - (*) reduct of a countable finitely bounded homogeneous structure [Bodirsky,Pinsker'11]

More on tractability conjecture
for finite structures

Polymorphism clone

- **Def:** $\mathrm{Pol}(\mathbb{A}) = \{f \mid f : \mathbb{A}^n \to \mathbb{A} \text{ homo}\}$ the set of polymorphisms
  - Polymorphisms generalize endomorphisms (automorphisms)
  - Polymorphisms are symmetries of higher arity
- **Fact:** $\mathrm{Pol}(\mathbb{A})$ is a clone: contains projections, closed under composition
- **Def:** Homomorphism between clones $\mathcal{A} \to \mathcal{B}$: mapping from operations in $\mathcal{A}$ to operations in $\mathcal{B}$ that
  - preserves projections and composition
  - equivalently, preserves (universally quantified) equations;
    eg. an associative/commutative operation is mapped to an associative/commutative operation

pp-interpretations and clone homomorphisms

- **Thm:** Finite $\mathbb{A}$ pp-interprets finite $\mathbb{B}$
  iff $\exists$ homo $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$
  - pp-definitions $\leftrightarrow$ polymorphisms [Geiger'68, Bodnarčuk, Kalužnin, Kotov, Romov'69]
  - pp-interpretations $\leftrightarrow$ standard algebraic constructions (powers, subalgebras, quotients) [Bodirsky, Willard]
  - standard constructions $\leftrightarrow$ equations (clone homomorphisms) [Birkhoff'35]
- Trivial clone $\mathcal{P}$: clone of projections on (say) 2-element set
  - a.k.a. 0, 1, 2
  - **Example:** Pol(3–SAT).
- **Corollary:** Finite $\mathbb{A}$ pp-interprets all finite
  iff $\exists$ homo $\mathrm{Pol}(\mathbb{A}) \to \mathcal{P}$

  ie. the set of equations satisfied by polymorphisms of $\mathbb{A}$ is satisfiable by projections

Tractability conjecture

Conjecture (Bulatov, Jeavons, Krokhin'00)

Assume $\mathbb{A}$ *finite.*
If $\nexists$ *homo* $\mathrm{Pol}(\mathbb{A}) \to \mathcal{P}$, *then* $\mathrm{CSP}(\mathbb{A})$ *in P.*

Recall: Otherwise $\mathrm{CSP}(\mathbb{A})$ is NP-complete.

Theorem

Assume $\mathbb{A}$ *finite, TFAE*

- $\mathbb{A}$ *does not pp-interpret all finite*
- $\nexists$ *homo* $\mathrm{Pol}(\mathbb{A}) \to \mathcal{P}$
  ie. polymorphisms satisfy nontrivial equations
- $\mathrm{Pol}(\mathbb{A})$ *contains an operation* s *of arity* 6 *such that*
  $s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$ *[Siggers'10]*

3rd item: concrete and positive alternative

Finite tractability conjecture vs. reality

Conjecture (Bulatov, Jeavons, Krokhin; Siggers)

Assume $\mathbb{A}$ finite. TFAE (assume $P \neq NP$)

- $\mathrm{CSP}(\mathbb{A})$ *in P*
- $\mathrm{Pol}(\mathbb{A})$ *contains an operation s of arity 6 such that*

$$s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$$

Reality can be worse:

Finite tractability conjecture vs. reality

Future theorem (T.B.D. 2048)

Assume $\mathbb{A}$ finite. TFAE

- $\mathrm{CSP}(\mathbb{A})$ *in P*
- $\mathrm{Pol}(\mathbb{A})$ *contains operations* $t_1, t_2, \ldots$ *such that*

$$t_1(x_3, x_1, x_2, x_2) = t_2(x_3, x_3, x_1)$$
$$t_{20}(x_1, x_1, x_2) = t_{13}(x_2, x_1)$$
$$\ldots$$

...but certainly the characterization looks somewhat like this
[Barto,Pinsker'?]
ie. depends only on height 1 equations
(could be a disjunction of such conditions, at worst)

More on tractability conjecture
for infinite structures

ω-categorical structures

- Every computational problem is P-time equivalent to $\mathrm{CSP}(\mathbb{A})$ [Bodirsky,Grohe'08]
- Some parts of the theory work well for countable ω-categorical structures
- Countable $\mathbb{A}$ is ω-categorical if
  - it is the unique countable model of its first order theory
  - equivalently, action of $\mathrm{Aut}(\mathbb{A})$ on n-tuples has finitely many orbits $\forall n$
- **Examples:** $(\mathbb{Q}, <)$, random graph, random poset
- CSPs over ω-categorical structures still essentially cover all computational problems (!) [Bodirsky,Grohe'08]

A reasonable class

- The Pinsker–Bodirsky infinite tractability conjecture concerns **reducts of finitely bounded homogeneous structures**
  - homogeneous = isomorphisms between finite induced substructures extend to automorphisms
  - finitely bounded = finite induced substructres can be characterized by finitely many forbidden substructures
  - reduct = has a first order definition in it
- they are ω-categorical
- their CSP is in NP
- **Examples:** $(\mathbb{Q}, <)$, random graph, random poset and reducts
- Complexity classified for these examples [Bodirsky,Kára'09],[Bodirsky,Pinsker'15],[Kompatscher,Van Pham]

pp-interpretations and clone homomorphisms

- Pol($\mathbb{A}$) carries a natural topology of pointwise convergence (metrizable)
- **Thm:** ω-categorical $\mathbb{A}$ pp-interprets finite $\mathbb{B}$
  iff $\exists$ **continuous** homo Pol($\mathbb{A}$) $\to$ Pol($\mathbb{B}$)
  - pp-definitions $\leftrightarrow$ polymorphisms [Bodirsky, Nešetřil'06]
  - pp-interpretations $\leftrightarrow$ **finite** powers, subalgebras, quotients
  - these constructions $\leftrightarrow$ continuous clone homomorphisms [Bodirsky, Pinsker'15]
- **Corollary:** ω-categorical $\mathbb{A}$ pp-interprets all finite
  iff $\exists$ continuous homo Pol($\mathbb{A}$) $\to \mathcal{P}$
  - how to translate it to human language?

Tractability conjecture

Conjecture (Bodirsky,Pinsker'11)

Assume $\mathbb{A}$ *reduct of finitely bounded homogeneous.*
If $\nexists$ *continuous homo* $\mathrm{Pol}(\mathbb{A}) \to \mathcal{P}$, *then* $\mathrm{CSP}(\mathbb{A})$ *in P.*

Recall: Otherwise $\mathrm{CSP}(\mathbb{A})$ is NP-complete.

Theorem

Assume $\mathbb{A}$ *reduct of . . . . TFAE*

- $\mathbb{A}$ *does not pp-interpret all finite*
- $\nexists$ *continuous homo* $\mathrm{Pol}(\mathbb{A}) \to \mathcal{P}$

Questions:

- is continuity important? (Why do we care?)
- is there a concrete positive alternative?

"Proof" of the conjecture

Conjecture (Bodirsky,Pinsker'11)

Assume $\mathbb{A}$ reduct of finitely bounded homogeneous.
If $\nexists$ continuous homo $\mathrm{Pol}(\mathbb{A}) \to \mathcal{P}$, then $\mathrm{CSP}(\mathbb{A})$ in P.

"Proof".

Assume $\nexists$ continuous homo $\mathrm{Pol}(\mathbb{A}) \to \mathcal{P}$

(1) $\nexists$ homo $\mathrm{Pol}(\mathbb{A}) \to \mathcal{P}$ ie. $\mathbb{A}$ has nontrivial polymorphisms

(2) WLOG $\mathbb{A}$ is a reduct of a homogeneous **Ramsey** structure

(3) From Ramsey $\Rightarrow$ some associated finite structure has non-trivial polymorphisms

(4) Use the algorithm from finite CSP and

(5) lift it to the original CSP

□

Gaps in all 5 stages! Progress on (5) by Bodirsky and Mottet

News (all positive)

- **Topology is irrelevant + positive alternative**
  Read: Barto, Pinsker: The algebraic dichotomy conjecture for infinite domain CSP, LICS'16
- **We have a better conjecture**
  Read: Barto, Opršal, Pinsker: The wonderland of reflections
- **We do not have a better conjecture**
  Read: Olšák: ?

(1) is done

Theorem (Barto, Pinsker'16)

Assume $\mathbb{A}$ *ω-categorical (includes reducts of ...). TFAE*

- $\mathbb{A}$ *does not pp-interpret all finite*
- $\nexists$ *continuous homo* $\mathrm{Pol}(\mathbb{A}) \to \mathcal{P}$
- $\nexists$ *homo* $\mathrm{Pol}(\mathbb{A}) \to \mathcal{P}$
- $\mathrm{Pol}(\mathbb{A})$ *contains unary* α, β *and 6-ary* s

$$\alpha(s(x, y, x, z, y, z)) = \beta(s(y, x, z, x, z, y))$$

Proof: Finite + compactness + extras

But: We can't claim that the complexity for sure only depends on equations

A better conjecture

New conjecture (Barto, Opršal, Pinsker)

Assume $\mathbb{A}$ *reduct of finitely bounded homogeneous.*
If $\nexists$ **uniformly** *continuous* **height 1** *homo* $\mathrm{Pol}(\mathbb{A}) \to \mathcal{P}$*, then* $\mathrm{CSP}(\mathbb{A})$ *in P.*

- Stronger topological requirement
- **But:** continuous homo exists iff uniformly continuous exists [Bodirsky,Pinsker'15]
- Weaker algebraic requirement (only height 1 equations)
- **Fact:** Otherwise $\mathrm{CSP}(\mathbb{A})$ is NP-complete.
- **So:** Old conjecture true $\Rightarrow$ new conjecture true

The two conjectures are equivalent

Theorem (Olšák)

The two conjectures are equivalent

- **Question:** Can we drop the (uniform) continuity from the new conjecture?

Conclusion

$>$ 20% done :)

Join the party until it's late!

Thank you!

Proof of pseudo-Siggers via a pseudo-loop lemma

Loop lemmata

Loop lemma: Let $R \subseteq B^2$, B finite.
Then R contains a loop (a, a) provided

- certain structural assumption is satisfied, like
  (1) R is symmetric and contains a triangle
  (2) R is symmetric and contains an odd cycle
  (3) R is strongly connected, GCD of cycles lengths $= 1$
  (4) R has no sources or sinks and has algebraic length 1
- and (B, R) does not pp-interpret, with parameters, $\mathbb{K}_3$

Each loop lemma gives equations [Siggers'10] using a standard universal algebraic argument [Kearnes,Marković,McKenzie'14]

Loop lemma proved by

- ∼[Hell, Nešetřil'90] assuming (2): purely relational proof
- [Bulatov'05] assuming (2): relations + operations
- [Barto, Kozik, Niven'09] assuming (4): purely algebraic proof

Equations from loop lemma

Theorem (Siggers'10)

TFAE for finite core $\mathbb{A}$

(1) *$\mathbb{A}$ does not pp-interpret $\mathbb{K}_3$ (with parameters).*

(2) *$\mathbb{A}$ has a polymorphism s satisfying*
$s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$.

Only (1) $\Rightarrow$ (2) interesting.

- Define $B = A^{A^3}$ (element is a 3-ary operation on A)
- Define

$$R = \left\{ \begin{pmatrix} (x,y,z) \mapsto s(x,y,x,z,y,z) \\ (x,y,z) \mapsto s(y,x,z,x,z,y) \end{pmatrix} \mid s \text{ a 6-ary polymorphism} \right\}$$

- Observe
  - R is symmetric
  - R contains a triangle (x, y, z form a triangle)
  - $\mathbb{A}$ pp-interprets $(B; R)$, thus $(B; R)$ does not interpret $\mathbb{K}_3$
- Loop in R gives the Siggers operation

To the infinity

Two issues with generalization to ω-categorical:

- A^{A^3} is not a finite power
  - Use A^X for X finite subsets of A^3
  - This gives "locally nice operations"
  - Compactness argument using ω-categoricity $\rightarrow$ globally nice operations
- loop lemma does not hold (eg. $(\mathbb{N}; \neq)$)
  - loop lemma $\rightarrow$ pseudo–loop lemma
  - pseudo–loop lemma: the technical core

We get:

Theorem

Let $\mathbb{A}$ be a core ω-categorical structure. TFAE

(1) *$\mathbb{A}$ does not pp-interpret $\mathbb{K}_3$ (with parameters).*

(2) *$\mathbb{A}$ has polymorphisms α, β, s satisfying* $\alpha s(x, y, x, z, y, z) = \beta s(y, x, z, x, z, y)$.

Pseudo–loop lemma

Theorem

Let $R \subseteq B^2$, B countable.
Let a group G acts on B oligomorphically, let R be G-invariant.
Then R contains a pseudo–loop (a, b), a, b in the same G-orbit provided

- *R is symmetric and contains a triangle*
- *and $(B; R, G\text{−orbits of pairs})$ does not pp-interpret, with parameters, $\mathbb{K}_3$*

- First attempt: use B,K,N algebraic approach. Not successful yet, but **very** interesting "side product" – Olšák's equations
- Second attempt: use Bulatov's relational/algebraic approach. Success, proof requires generalizations and extra work