Constraint Satisfaction Problems of Bounded Width

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CSP

Definition (CSP)

INPUT
- A: *set (of values)*
- X: *set (of variables)*
- $\mathcal{C}$: *set of constraints*

Constraint is a pair $C = (\vec{s}, R)$, *where*

- $\vec{s} = (x_1, \ldots, x_k)$: *k-tuple of vars* — *constraint scope*
- R: *k-ary relation on A, i.e.* $R \subseteq A^k$ — *constraint relation*

QUESTION *Is there a solution?*

Solution is a mapping $f : X \to A$ *satisfying all the constraints*

f *satisfies* $C = ((x_1, \ldots, x_k), R)$, *if* $(f(x_1), \ldots, f(x_k)) \in R$

CSP over a constraint language

Definition

Constraint language $= (A, \Gamma)$, *where* Γ *is a family of relations on* A *(sometimes* Γ *is assumed to be finite).*

Definition ($CSP(A, \Gamma)$)

INPUT	X: *set (of variables)* $\mathcal{C}$: *set of constraints over* (A, Γ)

Constraint over (A, Γ) *is a pair* $C = (\vec{s}, R)$, *where*

- $\vec{s} = (x_1, \ldots, x_k)$: *k-tuple of vars*
- R: *k-ary relation on* A, $R \in \Gamma$

QUESTION	*Is there a solution?*

Examples of $CSP(A, \Gamma)$

- SAT, 3-SAT, 2-SAT, HORN-SAT, ...
- Solving system of equations (for example linear equations over finite fields)
- 2-coloring, 3-coloring, ...
- homomorphism problems with fixed target structure
- ST-connectivity, ...
- practical problems (scheduling, ...)

The Holy Grail

Conjecture (Feder, Vardi 98)

For every finite constraint language (A, Γ), $CSP(A, \Gamma)$ is tractable or NP-complete.

Some evidence:

- Schaefer 78 True, if $|A| = 2$
- Bulatov 02 True, if $|A| = 3$
- Bulatov 03 True, if Γ contains all unary relations on A

The algebraic approach

Definition

n-ary operation on A = mapping $A^n \to A$
Algebra = pair (A, F), where F is a family of operations on A

To every constraint language (A, Γ) we assign algebra $Pol(A, \Gamma) = (A, F)$, where F are all operations compatible with all relations in Γ.

Theorem (Bulatov, Cohen, Gyssens, Jeavons, Krokhin 98-05)

The complexity of (A, Γ) depends only on $Pol(A, \Gamma)$.
(And much more...)

The algebraic dichotomy conjecture

Theorem (BJK 00-05)

If "there is a trivial algebra inside $Pol(A, \Gamma)$", then $CSP(A, \Gamma)$ is NP-complete.

Conjecture (BJK 05)

Otherwise, $CSP(A, \Gamma)$ is in P.

Theorem (Maróti, McKenzie 06)

Let (A, Γ) be a core constraint language. TFAE

- *"there is no trivial algebra inside $Pol(A, \Gamma)$"*
- *$Pol(A, \Gamma)$ contains a WNU operation f of some arity $k \geq 2$:*

$$f(a, a, \dots, a) = a$$

$$f(b, a, a, \dots, a) = f(a, b, a, a, \dots, a) = \dots = f(a, a, \dots, a, b)$$

Poly-time algorithms for CSPs

- Generalization of Gaussian elimination
  - Already well understood
  - Bulatov, Dalmau 06   Dalmau 06   Berman, Idziak, Marković, McKenzie, Valeriote, Willard 07
- Local Consistency Checking
  - The most natural family of algorithms for CSP
  - When can it be applied? . . . Larose and Zádori 07 conjecture
  - Crucial before attacking the dichotomy conjecture
  - Some partial results - Feder, Vardi 98, Dalmau, Pearson 99, Bulatov 06, Kiss, Valeriote 07, Carvalho, Dalmau, Marković, Maróti 09
  - Our theorem gives an affirmative answer

Minimal instance

Definition

Let $k \leq l$ be natural numbers.

An instance of CSP is called (k, l)-minimal, if

- *Every l-element set of vars is within a scope of some constraint*
- *For every set K of at most k variables and every pair of constraints $C_i = (\vec{s}_i, R_i)$ and $C_j = (\vec{s}_j, R_j)$ whose scopes contain K, the projections of R_i and R_j onto K are equal.*

- Every instance of *CSP* can be converted to a (k, l)-minimal instance with the same set of solutions in poly-time.
- If some (equivalently every) constraint relation is empty, then the original *CSP* has no solution.
- Otherwise, we don't know

Relational width

Definition

A constraint language (A, Γ) *has* *width* (k, l)*, if every instance of* $CSP(A, \Gamma)$*, such that the corresponding* (k, l)*-minimal instance has nonempty constraint relations, has a solution.*

A constraint language (A, Γ) *has* *bounded width**, if it has width* (k, l) *for some* k, l*.*

Many equivalent definitions (Datalog, bounded tree width duality, pebble games, . . .).

Theorem (Larose, Zádori 07)

If a core *constraint language* (A, Γ) *has bounded width, then "there is no module inside* $Pol(A, \Gamma)$*".*

Conjecture (Larose, Zádori)

The other implication is also true.

The theorem

Theorem (Maróti, McKenzie 06)

Let (A, Γ) be a core *constraint language. TFAE*

- *"there is no module inside Pol(A, Γ)"*
- *Pol(A, Γ) contains WNU operations of all but finitely many arities*

Theorem (Barto, Kozik 09)

- (A, Γ) *has bounded width*
- *new* (A, Γ) *has width* $(2, 3)$ *(optimal - Dalmau)*

Moreover, these conditions can be checked in poly-time.

Recently, a different proof announced by Bulatov!

Coffee!

ThANK you fOR your atTENTion?