Cyclic terms for join semi-distributive varieties II

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joint work with Libor Barto

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Cyclic algebras

Definition (Cyclic algebra)

Let $\mathbf{A}$ *be a finite algebra. A subalgebra* $\mathbf{R} \leq \mathbf{A}^n$ *is cyclic, if*

$$\forall a_1, \ldots, a_n \in A \qquad (a_1, a_2, \ldots, a_n) \in R \Rightarrow (a_2, \ldots, a_n, a_1) \in R$$

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Definition (A retraction)

For a relational structure (A, R) *a function* $f : A \to A$ *is a* retraction *iff*

- $f(f(a)) = f(a)$ *for all* $a \in A$ *and*
- *if* $(a_1, \ldots, a_n) \in R$ *then* $\left(f(a_1), \ldots, f(a_n)\right) \in R$ (*endomorphism*).

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and more generally

Theorem (General case)

Let $\mathbf{A}$ *be a finite, ~~simple~~ algebra from an SD(*$\vee$*) variety and let* p *be a prime number greater than* $|A|$ *and let* $\mathbf{R} \leq \mathbf{A}^p$ *be a cyclic ~~and subdirect~~ subalgebra of* $\mathbf{A}^p$*. Then* R *contains a constant tuple.*

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  - from minimality of **A** it contains a constant tuple ... OK

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thus $f_i = \mathrm{id}_A$ for $i \neq j$ and finally

$$(f_j, \mathrm{id}_A, \ldots, \mathrm{id}_A) \circ (\mathrm{id}_A, f_j, \mathrm{id}_A, \ldots, \mathrm{id}_A) \circ \cdots \circ (\mathrm{id}_A, \ldots, \mathrm{id}_A, f_j) = (f_j, \ldots, f_j)$$

The end

Thank you for your attention.