Constraint Satisfaction Problems of Bounded Width I

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Constraint Satisfaction Problem

Definition (A homomorphism of relational structures)

For two similar relational structures $\mathcal{R} = \{R; R_1, \dots, R_n\}$ *and* $\mathcal{S} = \{S; S_1, \dots S_n\}$ *a function* $h : R \to S$ *is a* *homomorphism* *iff*

$$(a_1, \dots, a_{n_i}) \in R_i \text{ implies } \big(h(a_1), \dots, h(a_{n_i})\big) \in S_i \text{ for any } i \leq n.$$

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A finite relational structure $\mathcal{R}$ *is a core if all the homomorphisms from* $\mathcal{R}$ *to* $\mathcal{R}$ *are bijective.*

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Definition (The combinatorial CSP)

For a fixed, finite relational structure $\mathcal{S}$ *by* CSP($\mathcal{S}$) *we understand a computational problem:*

INPUT: a relational structure $\mathcal{R}$ *similar to* $\mathcal{S}$

QUESTION: does there exist a homomorphism from $\mathcal{R}$ *to* $\mathcal{S}$*?*

Example I: Solving systems of linear equations over $\mathbb{Z}_2$

System of equations:

$$
\begin{aligned}
x + y &= z \\
x + z &= 0 \\
z + y &= 1
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Such a system has a solution if there is a function from $\{x, y, z, a, b\}$ to $\{0, 1\}$ such that

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- $a \mapsto 0$ and $b \mapsto 1$
- each triple (x, y, z), (x, z, a), (z, y, b) is mapped into the set

$$\left\{ \begin{array}{ll} (0,0,0), & (0,1,1) \\ (1,0,1), & (1,1,0) \end{array} \right\}$$

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Solving systems of linear equations over $\mathbb{Z}_2$ can be viewed as $\mathrm{CSP}(\{\{0,1\}, R_0, R_1, R_2\})$ where

$$\begin{array}{ll} R_0 & = \{0\} \\ R_1 & = \{1\} \end{array} \text{ and } R_2 = \left\{ \begin{array}{cc} (0,0,0), & (0,1,1) \\ (1,0,1), & (1,1,0) \end{array} \right\}$$

Example II: Coloring of undirected graphs

Fact (Two-coloring of undirected graphs)

Two-coloring of undirected graphs can be viewed as $\mathrm{CSP}(\mathcal{R})$ *where* $\mathcal{R}$ *is a relational structure over* $\{0,1\}$ *with one relation* $\{(0,1),(1,0)\}$.

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Two coloring of undirected graphs is solvable in polynomial time, while three coloring of undirected graphs is NP-complete.

Computational complexity of CSPs

Some CSPs are NP-complete and some are solvable in polynomial time...

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The most important algorithms solving some CSPs in polynomial time are:

- generalizations of Gaussian elimination: Dalmau's algorithm, algorithm for CSPs with few subpowers
- local consistency checking algorithms

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Consider vertices a,c and b

$$B_{bc} = \{ \quad (0,1), (1,0) \quad \}$$

For each $x \neq y \in \{a, b, c, d, e\}$ by B_{xy} we denote the set of pairs on 0, 1 such that:

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Consider vertices a,e and d

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Local consistency checking and strategies

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A (k, l)-strategy for relational structures $\mathcal{R}$ and $\mathcal{S}$ is a set $\mathbb{H}$ consisting of partial homomorphisms from $\mathcal{R}$ to $\mathcal{S}$ and such that

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For fixed k, l and $\mathcal{S}$ a maximal (k,l)-strategy for $\mathcal{R}$ and $\mathcal{S}$ can be found by local consistency checking in a time polynomial with respect to the size of $\mathcal{R}$.

Local consistency checking and CSPs of bounded width

Note that if h is a homomorphism from $\mathcal{R}$ to $\mathcal{S}$ then the set

$$\{h_{|A} \mid A \subseteq R \text{ and } |A| \leq l\}$$

is a (k, l)-strategy for $\mathcal{R}$ and $\mathcal{S}$ (for any $k \leq l$).

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A CSP($\mathcal{S}$) *(or simply $\mathcal{S}$) is of bounded width if there exists (k, l) such that for any $\mathcal{R}$ similar to $\mathcal{S}$ TFAE:*

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The structures of bounded width are those with CSP solvable by local consistency checking.

The algebraic approach

An m-ary operation t is compatible with an n-ary relation R if:

$$
\begin{array}{cccccccc}
t\,(& a_{11} & a_{12} & \dots & a_{1m} &) & = & a'_1 \\
t\,(& a_{21} & a_{22} & \dots & a_{2m} &) & = & a'_2 \\
 & \vdots & \vdots & \ddots & \vdots & & & \vdots \\
t\,(& a_{n1} & a_{n2} & \dots & a_{nm} &) & = & a'_n \\
 & & & \dots & & & & \\
 & \pitchfork & \pitchfork & & \pitchfork & & & \\
 & R & R & \dots & R & & &
\end{array}
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An operation t is a polymorphism of $\mathcal{S} = \{S; S_1, \ldots S_l\}$ if it is compatible with all the relations in $\mathcal{S}$.

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Theorem (from work of Cohen, Jeavons, Pearson, Bulatov, Krokhin)

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A finite core $\mathcal{S}$ is of bounded width if and only if the associated algebra **S** belongs to a congruence meet semi-distributive variety (is an SD($\wedge$) algebra), or equivalently (by work of Maróti and McKenzie) $\mathcal{S}$ has polymorphisms satisfying

$$w(x,\dots,x) = x \text{ and}$$
$$w(y,x,\dots,x) = w(x,y,x,\dots,x) = \dots = w(x,\dots,x,y)$$

of all but finitely many arities.

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Let $\mathcal{S}$ be a relational structure with a maximal arity of relation n and $\mathbb{H}$ be a (k, l)-strategy for $\mathcal{R}$ and $\mathcal{S}$. Any function h from $\mathcal{R}$ to $\mathcal{S}$ satisfying

$$h_{|A} \in \mathbb{H} \text{ for any } n\text{-element subset of the domain } A$$

is a homomorphism from $\mathcal{R}$ to $\mathcal{S}$.

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For a $(2k, 3k)$-strategy $\mathbb{H}$ compatible with an SD($\wedge$) algebra one can form a graph where

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and obtain an SD($\wedge$) algebra $\mathbf{A}$ together with

- $\mathbf{B}_i, i < n$: subalgebras of $\mathbf{A}$
- $\mathbf{B}_{ij}, i, j < n$: subalgebras of $\mathbf{B}_i \times \mathbf{B}_j$
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