Prime Maltsev Conditions

Libor Barto

joint work with Jakub Opršal

Charles University in Prague

NSAC 2013, June 7, 2013

Outline

- (Part 1) Interpretations
- (Part 2) Lattice of interpretability
- (Part 3) Prime filters
- (Part 4) Syntactic approach
- (Part 4) Relational approach

(Part 1)
Interpretations

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- the identity $m(x, y, y) \approx m(y, y, x) \approx x$
- $f : \mathcal{V} \to \mathcal{W}$ is determined by $m' = f(m)$
- m' must satisfy $m'(x, y, y) \approx m(y, y, x) \approx x$

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Example: Assume $\mathcal{V}$ is idempotent. No interpretation $\mathcal{V} \to \mathit{Sets}$ equivalent to the existence of a Taylor term in $\mathcal{V}$

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Birkhoff theorem $\Rightarrow$ $\forall$ interpretation is of the form $A \circ H \circ S \circ P$.

Interpretations are complicated

Theorem (B, 2006)

The category of varieties and interpretations is as complicated as it can be.

For instance: every small category is a full subcategory of it

(Part 2)
Lattice of Interpretability

Neumann 74

Garcia, Taylor 84

The lattice L

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- $\mathbf{A} \leq \mathbf{B}$ iff $\mathrm{Clo}(\mathbf{B}) \in AHSP\,\mathrm{Clo}(\mathbf{A})$

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$\mathbf{A} \wedge \mathbf{B}$ ($\mathbf{A}$ and $\mathbf{B}$ are clones)

Base set $= A \times B$
operations are $f \times g$, where f (resp. g) is an operation of $\mathbf{A}$ (resp. $\mathbf{B}$)

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- Many important theorems talk (indirectly) about (subposets of) L
  - Every nonzero locally finite idempotent variety is above a single nonzero variety Siggers
  - NU = EDGE $\cap$ CD (as filters) Berman, Idziak, Marković, McKenzie, Valeriote, Willard
  - no finite member of CD $\setminus$ NU is finitely related B

(Part 3)
Prime filters

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My motivation: Very basic syntactic question, close to the category theory I was doing, I should start with it

(Part 4)
Syntactic approach

Congruence permutable varieties

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Unfortunately

- The proof is complicated, long and technical
- Does not provide much insight
- Seems close to impossible to generalize

Coloring terms by variables

Definition (Segueira, (B))

Let A be a set of equivalences on X. We say that $\mathcal{V}$ is A-colorable, if there exists $c : F_{\mathcal{V}}(X) \to X$ such that $c(x) = x$ for all $x \in X$ and

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- If $\mathcal{V}$ has a Maltsev term then it is not A-colorable
- The converse is also true

Coloring continued

- $\mathcal{V}$ is congruence permutable iff $\mathcal{V}$ is A-colorable for $A = \ldots$
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Results coming from this notion Sequeira, Bentz, Opršal, (B):

- The join of two varieties which are **linear** and not congruence permutable/n-permutable/modular is not congruence permutable/ . . .
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- $+$ proofs are simple and natural
- - works (so far) only for linear identities

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Open problem: For some natural class of filters, is it true that F is prime iff members of F can be described by A-colorability for some A?

(Part 5)
Relational approach

(pp)-interpretation between relational structures

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- pp-definitions
- induced substructures on a pp-definable subsets
- Cartesian powers of structures
- other powers

Results

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Theorem

If $\mathcal{V}$, $\mathcal{W}$ are not permutable/n-permutable for some n/modular and (*) then neither is $\mathcal{V} \vee \mathcal{W}$

- (*) = locally finite idempotent
- for n-permutability (*) = locally finite, or (*) = idempotent Valeriote, Willard
- for modularity, it follows form the work of McGarry, Valeriote