Finitely related algebras

Libor Barto

Charles University in Prague

SSAOS 2013, September 5, 2013

Topic

Fact: Every algebra (clone) on a finite set is determined by a set of relations.

Topic

Fact: Every algebra (clone) on a finite set is determined by a set of relations.

Question: When can be this set of relation chosen finite?

Polymorphisms

$\mathbf{A}$ ... Algebra with finite universe A
$\mathrm{Clo}(\mathbf{A})$ ... Clone of $\mathbf{A}$

Polymorphisms

$\mathbf{A}$ ... Algebra with finite universe A
$\mathrm{Clo}(\mathbf{A})$ ... Clone of $\mathbf{A}$

$\mathbb{A}$ ... Relational structure with finite universe A
$\mathrm{Pol}(\mathbb{A})$ ... Clone of polymorphisms of $\mathbb{A}$

Polymorphisms

$\mathbf{A}$ ... Algebra with finite universe A
Clo($\mathbf{A}$) ... Clone of $\mathbf{A}$

$\mathbb{A}$ ... Relational structure with finite universe A
Pol($\mathbb{A}$) ... Clone of polymorphisms of $\mathbb{A}$

Definition

$f \in \mathrm{Pol}(\mathbb{A})$ if for every relation R in $\mathbb{A}$
$\mathbf{a}_1, \ldots, \mathbf{a}_n \in R \Rightarrow f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \in R$

$$f \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} = \begin{pmatrix} f(a_{11}, \ldots, a_{1n}) \\ f(a_{21}, \ldots, a_{2n}) \\ \vdots \\ f(a_{m1}, \ldots, a_{mn}) \end{pmatrix}$$

Columns in $R \Rightarrow$ result in R

Finitely related algebras

Theorem (Geiger'68; Bodnarčuk, Kalužnin, Kotov, Romov'69)

$\forall \mathbf{A} \quad \exists \mathbb{A} \quad \mathsf{Clo}(\mathbf{A}) = \mathsf{Pol}(\mathbb{A})$

Finitely related algebras

Theorem (Geiger'68; Bodnarčuk, Kalužnin, Kotov, Romov'69)

$\forall \mathbf{A} \quad \exists \mathbb{A} \quad \mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$

Proof.

- $\mathbb{A} = (A; F_1, F_2, \dots)$, where
  $F_m =$ all m-ary operations $\subset A^{A^m}$ viewed as A_m-ary relation

□

Finitely related algebras

Theorem (Geiger'68; Bodnarčuk, Kalužnin, Kotov, Romov'69)

$\forall \mathbf{A} \quad \exists \mathbb{A} \quad \mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$

Proof.

- $\mathbb{A} = (A; F_1, F_2, \dots)$, where
  $F_m =$ all m-ary operations $\subset A^{A^m}$ viewed as A_m-ary relation
- $\mathrm{Clo}(\mathbf{A}) \subseteq \mathrm{Pol}(\mathbb{A})$ because
  $f(g_1, \dots, g_n)$ (g_i's viewed as tuples of length A^m)
  $= f(g_1, \dots, g_n)$ (viewed as composition of operations)

□

Finitely related algebras

Theorem (Geiger'68; Bodnarčuk, Kalužnin, Kotov, Romov'69)

$\forall \mathbf{A} \quad \exists \mathbb{A} \quad \mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$

Proof.

- $\mathbb{A} = (A; F_1, F_2, \dots)$, where
  $F_m =$ all m-ary operations $\subset A^{A^m}$ viewed as A_m-ary relation
- $\mathrm{Clo}(\mathbf{A}) \subseteq \mathrm{Pol}(\mathbb{A})$ because
  $f(g_1, \dots, g_n)$ (g_i's viewed as tuples of length A^m)
  $= f(g_1, \dots, g_n)$ (viewed as composition of operations)
- $\mathrm{Clo}(\mathbf{A}) \supseteq \mathrm{Pol}(\mathbb{A})$ because $f(\pi_1, \dots, \pi_n) \in F_n$ means
  $f \in \mathrm{Clo}(\mathbf{A})$

□

Finitely related algebras

Theorem (Geiger'68; Bodnarčuk, Kalužnin, Kotov, Romov'69)

$\forall \mathbf{A} \quad \exists \mathbb{A} \quad \mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$

Definition

$\mathbf{A}$ is finitely related, if $\exists \mathbb{A}$ with finitely many relations such that $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$.

Finitely related algebras

Theorem (Geiger'68; Bodnarčuk, Kalužnin, Kotov, Romov'69)

$\forall \mathbf{A} \quad \exists \mathbb{A} \quad \mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$

Definition

$\mathbf{A}$ is finitely related, if $\exists \mathbb{A}$ with finitely many relations such that $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$.

Observation: Every clone is an intersection of a descending chain of finitely related clones:

$\mathrm{Pol}(A; F_1) \supseteq \mathrm{Pol}(A; F_1, F_2) \supseteq \dots$

Finitely related algebras - examples

- $\mathbf{A} = (\{0,1\};\ \wedge, \neg)$

  $\mathrm{Clo}(\mathbf{A}) =$ all operations
  $= \mathrm{Pol}(\{0,1\};$ no relations$)$

Finitely related algebras - examples

- $\mathbf{A} = (\{0,1\};\ \wedge, \neg)$

  $\text{Clo}(\mathbf{A})$ = all operations
  $= \text{Pol}(\{0,1\};$ no relations$)$

- $\mathbf{A} = (\{0,1\};$ no operations$)$

  $\text{Clo}(\mathbf{A})$ = projections
  $= \text{Pol}(\{0,1\};$ all relations$)$
  $= \text{Pol}(\{0,1\},$ (ternary) graph of NAND$)$

Finitely related algebras - examples

- $\mathbf{A} = (\{0,1\};\ \wedge, \neg)$

  $\mathrm{Clo}(\mathbf{A}) =$ all operations
  $= \mathrm{Pol}(\{0,1\};$ no relations$)$

- $\mathbf{A} = (\{0,1\};$ no operations$)$

  $\mathrm{Clo}(\mathbf{A}) =$ projections
  $= \mathrm{Pol}(\{0,1\};$ all relations$)$
  $= \mathrm{Pol}(\{0,1\},$ (ternary) graph of NAND$)$

- $\mathbf{A}$ has a near unanimity operations $\Rightarrow$ $\mathbf{A}$ is finitely related
  Baker, Pixley'75

Non-finitely related algebras

f ... n-ary operation

$f(x_1, x_1, x_2, \ldots, x_{n-1})$, $f(x_1, x_2, x_1, x_3, \ldots, x_{n-1})$, etc.
... identification minors

Non-finitely related algebras

f ... n-ary operation

$f(x_1, x_1, x_2, \dots, x_{n-1})$, $f(x_1, x_2, x_1, x_3, \dots, x_{n-1})$, etc.
... identification minors

Fact

$\mathbf{A}$ is finitely related iff $\exists n$ such that $\forall f$ of arity $\geq n$ every identification minor of f is in $\mathrm{Clo}(\mathbf{A}) \Rightarrow f \in \mathrm{Clo}(\mathbf{A})$

□

Non-finitely related algebras

f … n-ary operation

$f(x_1, x_1, x_2, \ldots, x_{n-1})$, $f(x_1, x_2, x_1, x_3, \ldots, x_{n-1})$, etc.
… identification minors

Fact

A *is finitely related iff* $\exists n$ *such that* $\forall f$ *of arity* $\geq n$
every identification minor of f *is in* $\mathrm{Clo}(\mathbf{A}) \Rightarrow f \in \mathrm{Clo}(\mathbf{A})$

Proof.

Assume

- $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; G)$, where $G \subseteq A^2$

□

Non-finitely related algebras

f ... n-ary operation

$f(x_1, x_1, x_2, \dots, x_{n-1})$, $f(x_1, x_2, x_1, x_3, \dots, x_{n-1})$, etc.
... identification minors

Fact

$\mathbf{A}$ is finitely related iff $\exists n$ such that $\forall f$ of arity $\geq n$ every identification minor of f is in $\mathrm{Clo}(\mathbf{A}) \Rightarrow f \in \mathrm{Clo}(\mathbf{A})$

Proof.

Assume

- $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; G)$, where $G \subseteq A^2$
- $n \geq |A|^2$
- f is n-ary operation whose $\forall$ identification minor is in $\mathrm{Clo}(\mathbf{A})$

□

Non-finitely related algebras

f ... n-ary operation

$f(x_1, x_1, x_2, \ldots, x_{n-1})$, $f(x_1, x_2, x_1, x_3, \ldots, x_{n-1})$, etc.
... identification minors

Fact

$\mathbf{A}$ is finitely related iff $\exists n$ such that $\forall f$ of arity $\geq n$ every identification minor of f is in $\mathrm{Clo}(\mathbf{A}) \Rightarrow f \in \mathrm{Clo}(\mathbf{A})$

Proof.

Assume

- $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; G)$, where $G \subseteq A^2$
- $n \geq |A|^2$
- f is n-ary operation whose $\forall$ identification minor is in $\mathrm{Clo}(\mathbf{A})$

We want to show that $f \in \mathrm{Clo}(\mathbf{A})$, i.e. f preserves G

□

Non-finitely related algebras

f ... n-ary operation

$f(x_1, x_1, x_2, \dots, x_{n-1})$, $f(x_1, x_2, x_1, x_3, \dots, x_{n-1})$, etc.

... identification minors

Fact

$\mathbf{A}$ *is finitely related iff* $\exists n$ *such that* $\forall f$ *of arity* $\geq n$ *every identification minor of* f *is in* $\mathrm{Clo}(\mathbf{A}) \Rightarrow f \in \mathrm{Clo}(\mathbf{A})$

Proof.

Assume

- $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; G)$, where $G \subseteq A^2$
- $n \geq |A|^2$
- f is n-ary operation whose $\forall$ identification minor is in $\mathrm{Clo}(\mathbf{A})$

We want to show that $f \in \mathrm{Clo}(\mathbf{A})$, i.e. f preserves G

$$f\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & \dots & a_n \\ b_1 & b_2 & b_3 & b_4 & b_5 & \dots & b_n \end{pmatrix} \in G?$$

□

Non-finitely related algebras

f ... n-ary operation

$f(x_1, x_1, x_2, \ldots, x_{n-1})$, $f(x_1, x_2, x_1, x_3, \ldots, x_{n-1})$, etc.

... identification minors

Fact

A *is finitely related iff* $\exists n$ *such that* $\forall f$ *of arity* $\geq n$ *every identification minor of* f *is in* $\mathrm{Clo}(\mathbf{A}) \Rightarrow f \in \mathrm{Clo}(\mathbf{A})$

Proof.

Assume

- $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; G)$, where $G \subseteq A^2$
- $n \geq |A|^2$
- f is n-ary operation whose $\forall$ identification minor is in $\mathrm{Clo}(\mathbf{A})$

We want to show that $f \in \mathrm{Clo}(\mathbf{A})$, i.e. f preserves G

$$f\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & \ldots & a_n \\ b_1 & b_2 & b_3 & b_4 & b_5 & \ldots & b_n \end{pmatrix} \in G?$$

Yes, we can find two equal columns and use a minor □

Examples of non-finitely related algebras

Examples of non-finitely related algebras

1. $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\};\ \{0,1\}^n \setminus \{(0,0,\ldots,0)\}, n \in N) = \mathrm{Clo}(\{0,1\};\ \rightarrow)$

Examples of non-finitely related algebras

1. $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\};\ \{0,1\}^n \setminus \{(0,0,\ldots,0)\}, n \in N) = \mathrm{Clo}(\{0,1\};\ \rightarrow)$

   $f \in \mathrm{Clo}(\mathbf{A})$ iff $\exists i$ such that $a_i = 1 \Rightarrow f(a_1,\ldots,a_n) = 1$

Examples of non-finitely related algebras

1. $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\};\ \{0,1\}^n \setminus \{(0,0,\ldots,0)\}, n \in N) = \mathrm{Clo}(\{0,1\};\ \rightarrow)$

   $f \in \mathrm{Clo}(\mathbf{A})$ iff $\exists i$ such that $a_i = 1 \Rightarrow f(a_1,\ldots,a_n) = 1$

   - Take any $n \in N$ and define
     $f(a_1,\ldots,a_n) = 1$ iff at least two 1s

Examples of non-finitely related algebras

1. $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\};\ \{0,1\}^n \setminus \{(0,0,\dots,0)\}, n \in N) = \mathrm{Clo}(\{0,1\};\ \rightarrow)$

   $f \in \mathrm{Clo}(\mathbf{A})$ iff $\exists i$ such that $a_i = 1 \Rightarrow f(a_1,\dots,a_n) = 1$

   - Take any $n \in N$ and define $f(a_1,\dots,a_n) = 1$ iff at least two 1s
   - $f \notin \mathrm{Clo}(\mathbf{A})$

Examples of non-finitely related algebras

1. $\text{Clo}(\mathbf{A}) = \text{Pol}(\{0,1\};\ \{0,1\}^n \setminus \{(0,0,\dots,0)\}, n \in N) = \text{Clo}(\{0,1\};\ \rightarrow)$

   $f \in \text{Clo}(\mathbf{A})$ iff $\exists i$ such that $a_i = 1 \Rightarrow f(a_1,\dots,a_n) = 1$

   - Take any $n \in N$ and define $f(a_1,\dots,a_n) = 1$ iff at least two 1s
   - $f \notin \text{Clo}(\mathbf{A})$
   - Every identification minor is in $\text{Clo}(\mathbf{A})$

Examples of non-finitely related algebras

1. $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\};\ \{0,1\}^n \setminus \{(0,0,\ldots,0)\}, n \in N) = \mathrm{Clo}(\{0,1\};\ \rightarrow)$

   $f \in \mathrm{Clo}(\mathbf{A})$ iff $\exists i$ such that $a_i = 1 \Rightarrow f(a_1,\ldots,a_n) = 1$

   - Take any $n \in N$ and define $f(a_1,\ldots,a_n) = 1$ iff at least two 1s
   - $f \notin \mathrm{Clo}(\mathbf{A})$
   - Every identification minor is in $\mathrm{Clo}(\mathbf{A})$
   - $\Rightarrow$ **A** is not finitely related

Examples of non-finitely related algebras

1. $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\};\ \{0,1\}^n \setminus \{(0,0,\ldots,0)\}, n \in N) = \mathrm{Clo}(\{0,1\};\ \rightarrow)$

   $f \in \mathrm{Clo}(\mathbf{A})$ iff $\exists i$ such that $a_i = 1 \Rightarrow f(a_1,\ldots,a_n) = 1$

2. **idempotent version**
   $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\};\{0\},\{1\},\{0,1\}^n \setminus \{(0,0,\ldots,0)\}, n \in N)$

Examples of non-finitely related algebras

1. $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\};\ \{0,1\}^n \setminus \{(0,0,\dots,0)\}, n \in N) = \mathrm{Clo}(\{0,1\};\ \rightarrow)$

   $f \in \mathrm{Clo}(\mathbf{A})$ iff $\exists i$ such that $a_i = 1 \Rightarrow f(a_1,\dots,a_n) = 1$

2. **idempotent version**
   $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\}; \{0\}, \{1\}, \{0,1\}^n \setminus \{(0,0,\dots,0)\}, n \in N)$

3. **more generally**, take A, $C \subsetneq D \subseteq A$
   $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; D^n \setminus (D \setminus C)^n, n \in N)$

   $f \in \mathrm{Clo}(\mathbf{A})$ iff $\exists i$ such that if $a_1,\dots,a_n \in D$ and $a_i \in C$ then $f(a_1,\dots,a_n) \in C$

Examples of non-finitely related algebras

1. $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\};\ \{0,1\}^n \setminus \{(0,0,\dots,0)\}, n \in N) = \mathrm{Clo}(\{0,1\};\ \rightarrow)$

   $f \in \mathrm{Clo}(\mathbf{A})$ iff $\exists i$ such that $a_i = 1 \Rightarrow f(a_1,\dots,a_n) = 1$

2. **idempotent version**
   $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\}; \{0\}, \{1\}, \{0,1\}^n \setminus \{(0,0,\dots,0)\}, n \in N)$

3. **more generally**, take A, $C \subsetneq D \subseteq A$
   $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; D^n \setminus (D \setminus C)^n, n \in N)$

   $f \in \mathrm{Clo}(\mathbf{A})$ iff $\exists i$ such that if $a_1,\dots,a_n \in D$ and $a_i \in C$ then $f(a_1,\dots,a_n) \in C$

4. **idempotent version**
   $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$

Examples of non-finitely related algebras

1. $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\};\ \{0,1\}^n \setminus \{(0,0,\ldots,0)\}, n \in N) = \mathrm{Clo}(\{0,1\};\ \rightarrow)$

   $f \in \mathrm{Clo}(\mathbf{A})$ iff $\exists i$ such that $a_i = 1 \Rightarrow f(a_1,\ldots,a_n) = 1$

2. **idempotent version**
   $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\{0,1\};\{0\},\{1\},\{0,1\}^n \setminus \{(0,0,\ldots,0)\}, n \in N)$

3. **more generally**, take A, $C \subsetneq D \subseteq A$
   $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; D^n \setminus (D \setminus C)^n, n \in N)$

   $f \in \mathrm{Clo}(\mathbf{A})$ iff $\exists i$ such that if $a_1,\ldots,a_n \in D$ and $a_i \in C$ then $f(a_1,\ldots,a_n) \in C$

4. **idempotent version**
   $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$

5. Not many examples this way... (should be uncountably many!)

Cube term blockers

This slide (and some other slides) . . . all algebras idempotent

Cube term blockers

This slide (and some other slides) ... all algebras idempotent

Definition

$\mathbf{A}$ algebra, $C \leq D \leq \mathbf{A}$ ($C \neq D$) is a cube term blocker, if
$\forall f \in \mathrm{Clo}(\mathbf{A}) \;\; \exists i \;\; f(D, D, \ldots, D, \underbrace{C}_{i}, D, \ldots, D) \subseteq C$

Cube term blockers

This slide (and some other slides) ... all algebras idempotent

Definition

$\mathbf{A}$ algebra, $C \leq D \leq \mathbf{A}$ $(C \neq D)$ is a cube term blocker, if
$\forall f \in \mathrm{Clo}(\mathbf{A})\ \ \exists i\ f(D, D, \ldots, D, \underbrace{C}_{i}, D, \ldots, D) \subseteq C$

Equivalent to: Every $f \in \mathrm{Clo}(\mathbf{A})$ preserves $D^m \setminus (D \setminus C)^m$.

Cube term blockers

This slide (and some other slides) ... all algebras idempotent

Definition

$\mathbf{A}$ algebra, $C \leq D \leq \mathbf{A}$ $(C \neq D)$ is a cube term blocker, if
$\forall f \in \mathrm{Clo}(\mathbf{A}) \ \ \exists i \ f(D, D, \ldots, D, \underbrace{C}_{i}, D, \ldots, D) \subseteq C$

Equivalent to: Every $f \in \mathrm{Clo}(\mathbf{A})$ preserves $D^m \setminus (D \setminus C)^m$.

Previous examples: Maximal clones containing a blocker

Cube term blockers

This slide (and some other slides) . . . all algebras idempotent

Definition

$\mathbf{A}$ algebra, $C \leq D \leq \mathbf{A}$ ($C \neq D$) is a cube term blocker, if
$\forall f \in \mathrm{Clo}(\mathbf{A}) \;\; \exists i \;\; f(D, D, \dots, D, \underbrace{C}_{i}, D, \dots, D) \subseteq C$

Equivalent to: Every $f \in \mathrm{Clo}(\mathbf{A})$ preserves $D^m \setminus (D \setminus C)^m$.

Previous examples: Maximal clones containing a blocker

Theorem (Marković, Maróti, McKenzie'12; B, Kozik, Stanovský)

$\mathbf{A}$ has no cube term blocker iff $\mathbf{A}$ has a cube term, i.e. a term operation t t satisfying some identities of the form

$t(x, ?, ? \dots, ?) = y, \;\; \dots, \;\; t(?, ?, \dots, ?, x) = y$

Cube term blockers

This slide (and some other slides) ... all algebras idempotent

Definition

$\mathbf{A}$ algebra, $C \leq D \leq \mathbf{A}$ ($C \neq D$) is a cube term blocker, if
$\forall f \in \mathrm{Clo}(\mathbf{A})\ \ \exists i\ f(D, D, \dots, D, \underbrace{C}_{i}, D, \dots, D) \subseteq C$

Equivalent to: Every $f \in \mathrm{Clo}(\mathbf{A})$ preserves $D^m \setminus (D \setminus C)^m$.

Previous examples: Maximal clones containing a blocker

Theorem (Marković, Maróti, McKenzie'12; B, Kozik, Stanovský)

$\mathbf{A}$ *has no cube term blocker iff* $\mathbf{A}$ *has a cube term, i.e. a term operation t t satisfying some identities of the form*

$t(x, ?, ? \dots, ?) = y,\ \dots,\ t(?, ?, \dots, ?, x) = y$

Examples: near unanimity operation, Mal'tsev operation

Cube terms, cntd.

Theorem (Aichinger, Mayr, McKenzie)

If $\mathbf{A}$ *has a cube term then* $\mathbf{A}$ *is finitely related.*

Consequences:

Cube terms, cntd.

Theorem (Aichinger, Mayr, McKenzie)

If $\mathbf{A}$ *has a cube term then* $\mathbf{A}$ *is finitely related.*

Consequences:

- Maximal non-finitely related idempotent clones are precisely $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$

Cube terms, cntd.

Theorem (Aichinger, Mayr, McKenzie)

If $\mathbf{A}$ *has a cube term then* $\mathbf{A}$ *is finitely related.*

Consequences:

- Maximal non-finitely related idempotent clones are precisely

  $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$

- An idempotent clone is upward inherently finitely related (every larger idempotent clone is finitely related) iff it has a cube term

CD $\Rightarrow$ NU

$\text{Clo}(\mathbf{A}) = \text{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$

Why is **A** not finitely related?

CD $\Rightarrow$ NU

$$\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$$

Why is **A** not finitely related?

Theorem (B'13, Zhuk)

If **A** *has Jónsson terms and does not have a near unanimity term then* **A** *is not finitely related*

CD $\Rightarrow$ NU

$\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$

Why is **A** not finitely related?

Theorem (B'13, Zhuk)

*If **A** has Jónsson terms and does not have a near unanimity term then **A** is not finitely related*

- **Directed Jónsson terms** (equivalent to Jónsson terms Kozik):
  $x \approx p_0(x, y, z), z \approx p_n(x, y, z)$
  $p_i(x, y, y) \approx p_{i+1}(x, x, y)$
  $p_i(x, y, x) \approx x$

CD $\Rightarrow$ NU

$\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$

Why is **A** not finitely related?

Theorem (B'13, Zhuk)

If **A** *has Jónsson terms and does not have a near unanimity term then* **A** *is not finitely related*

- **Directed Jónsson terms** (equivalent to Jónsson terms Kozik):
  $x \approx p_0(x, y, z), z \approx p_n(x, y, z)$
  $p_i(x, y, y) \approx p_{i+1}(x, x, y)$
  $p_i(x, y, x) \approx x$
- **Near unanimity term**: $t(x, \dots, x, y, x, \dots, x) \approx x$

Absorptions

$\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$

Why **A** is not finitely related? More general theorem:

Absorptions

$\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$

Why $\mathbf{A}$ is not finitely related? More general theorem:

Theorem (B, Bulín; Kozik)

If $\exists B\ B \triangleleft_j \mathbf{A}$ and $B \not\triangleleft \mathbf{A}$ then $\mathbf{A}$ is not finitely related

Absorptions

$\text{Clo}(\mathbf{A}) = \text{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$

Why $\mathbf{A}$ is not finitely related? More general theorem:

Theorem (B, Bulín; Kozik)

If $\exists B\ B \triangleleft_j \mathbf{A}$ and $B \ntriangleleft \mathbf{A}$ then $\mathbf{A}$ is not finitely related

- $B \triangleleft_j \mathbf{A}$ if $B \leq \mathbf{A}$ and
  $x \approx p_0(x, y, z), z \approx p_n(x, y, z)$
  $p_i(x, y, y) \approx p_{i+1}(x, x, y)$
  $p_i(B, A, B) \approx B$

  ($\mathbf{A}$ has Jónsson terms iff $\forall a\ \{a\} \triangleleft_j \mathbf{A}$)

Absorptions

$\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$

Why $\mathbf{A}$ is not finitely related? More general theorem:

Theorem (B, Bulín; Kozik)

If $\exists B\ B \triangleleft_j \mathbf{A}$ and $B \not\triangleleft \mathbf{A}$ then $\mathbf{A}$ is not finitely related

- $B \triangleleft_j \mathbf{A}$ if $B \leq \mathbf{A}$ and
  $x \approx p_0(x, y, z), z \approx p_n(x, y, z)$
  $p_i(x, y, y) \approx p_{i+1}(x, x, y)$
  $p_i(B, A, B) \approx B$

  ($\mathbf{A}$ has Jónsson terms iff $\forall a\ \{a\} \triangleleft_j \mathbf{A}$)

- $B \triangleleft \mathbf{A}$ if $B \leq \mathbf{A}$ and $t(B, \ldots, B, A, B, \ldots, B) \subseteq B$

  ($\mathbf{A}$ has a near unanimity term iff $\forall a\ \{a\} \triangleleft \mathbf{A}$)

Absorptions

$\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(A; \{a\}, a \in A,\ D^n \setminus (D \setminus C)^n, n \in N)$

Why **A** is not finitely related? More general theorem:

Theorem (B, Bulín; Kozik)

If $\exists B\ B \lhd_j \mathbf{A}$ and $B \not\lhd \mathbf{A}$ then $\mathbf{A}$ is not finitely related

- $B \lhd_j \mathbf{A}$ if $B \leq \mathbf{A}$ and
  $x \approx p_0(x, y, z), z \approx p_n(x, y, z)$
  $p_i(x, y, y) \approx p_{i+1}(x, x, y)$
  $p_i(B, A, B) \approx B$

  ($\mathbf{A}$ has Jónsson terms iff $\forall a\ \{a\} \lhd_j \mathbf{A}$)

- $B \lhd \mathbf{A}$ if $B \leq \mathbf{A}$ and $t(B, \ldots, B, A, B, \ldots, B) \subseteq B$

  ($\mathbf{A}$ has a near unanimity term iff $\forall a\ \{a\} \lhd \mathbf{A}$)

- Always $B \lhd \mathbf{A} \Rightarrow B \lhd_j \mathbf{A}$

"Give up your selfishness, and you shall find peace; like water mingling with water, you shall merge in absorption."

Sri Guru Granth Sahib

Weird Guess

Weird Guess

Let $\mathbf{A}$ *be idempotent. Then* $\mathbf{A}$ *is finitely related iff*
(*) $\forall B \; B \triangleleft_j \mathbf{A} \Rightarrow B \triangleleft \mathbf{A}$.

Weird Guess

Weird Guess

Let $\mathbf{A}$ *be idempotent. Then* $\mathbf{A}$ *is finitely related iff*
(*) $\forall B \;\; B \lhd_j \mathbf{A} \Rightarrow B \lhd \mathbf{A}$.

- $\Rightarrow$ is true

Weird Guess

Weird Guess

Let **A** *be idempotent. Then* **A** *is finitely related iff*
(*) $\forall B \; B \triangleleft_j \mathbf{A} \Rightarrow B \triangleleft \mathbf{A}$.

- $\Rightarrow$ is true
- If this is true then we can decide whether **A** is finitely related

Weird Guess

Weird Guess

Let $\mathbf{A}$ *be idempotent. Then* $\mathbf{A}$ *is finitely related iff*
(*) $\forall B \; B \lhd_j \mathbf{A} \Rightarrow B \lhd \mathbf{A}$.

- $\Rightarrow$ is true
- If this is true then we can decide whether $\mathbf{A}$ is finitely related
- True if $\mathbf{A}$ has a cube term, because

Weird Guess

Weird Guess

Let $\mathbf{A}$ *be idempotent. Then* $\mathbf{A}$ *is finitely related iff*
(*) $\forall B\ B \triangleleft_j \mathbf{A} \Rightarrow B \triangleleft \mathbf{A}$.

- $\Rightarrow$ is true
- If this is true then we can decide whether $\mathbf{A}$ is finitely related
- True if $\mathbf{A}$ has a cube term, because

Theorem (B, Kazda)

If $\mathbf{A}$ *has a cube term then (*) is true*

How to falsify Weird Guess I

Theorem (B)

If $\mathbf{A}$ *generates a congruence modular variety and does not have a cube term then* $\mathbf{A}$ *is not finitely related.*

How to falsify Weird Guess I

Theorem (B)

If **A** *generates a congruence modular variety and does not have a cube term then* **A** *is not finitely related.*

Does it violate Weird Guess???

In particular:

How to falsify Weird Guess I

Theorem (B)

*If **A** generates a congruence modular variety and does not have a cube term then **A** is not finitely related.*

Does it violate Weird Guess???

In particular:

Problem

*Find an idempotent algebra **A** such that*

- **A** *generates a congruence modular variety*
- **A** *does not have a cube term*
- **A** *has no proper Jónsson absorbing subuniverses*

How to falsify Weird Guess II

If Weird Guess is true then, for fixed A, there cannot be arbitrarily large chains of clones on A

$C_1 \subseteq C_2 \subseteq C_3 \subseteq \dots$

such that C_{even} are finitely related and C_{odd} are not

Problem

Find such long chains of idempotent clones!

How to falsify Weird Guess III

Every clone $C \subseteq \mathrm{Pol}$(all graphs of bijections) satisfy (*).

Problem

Are all such clones finitely related?

(True if C omits type 1)

Fininite relatedness and constructions - positive results

Fact

Let $\mathbf{B} \leq_{sd} \mathbf{A}^n$*. Then* $\mathbf{B}$ *is finitely related iff* $\mathbf{A}$ *is finitely related*

Fininite relatedness and constructions - positive results

Fact

Let $\mathbf{B} \leq_{sd} \mathbf{A}^n$*. Then* $\mathbf{B}$ *is finitely related iff* $\mathbf{A}$ *is finitely related*

Example: Every semilattice is finitely related

Fininite relatedness and constructions - positive results

Fact

Let $\mathbf{B} \leq_{sd} \mathbf{A}^n$*. Then* $\mathbf{B}$ *is finitely related iff* $\mathbf{A}$ *is finitely related*

Example: Every semilattice is finitely related

Example: If $\mathrm{HSP}(\mathbf{C}) = \mathrm{HSP}(\mathbf{D})$ then $\mathbf{C}$ is finitely related iff $\mathbf{D}$ is finitely related

Fininite relatedness and constructions - positive results

Fact

Let $\mathbf{B} \leq_{sd} \mathbf{A}^n$*. Then* $\mathbf{B}$ *is finitely related iff* $\mathbf{A}$ *is finitely related*

Example: Every semilattice is finitely related

Example: If $\mathsf{HSP}(\mathbf{C}) = \mathsf{HSP}(\mathbf{D})$ then $\mathbf{C}$ is finitely related iff $\mathbf{D}$ is finitely related

Partial reduction to the idempotent case:

Fininite relatedness and constructions - positive results

Fact

Let $\mathbf{B} \leq_{sd} \mathbf{A}^n$. *Then* $\mathbf{B}$ *is finitely related iff* $\mathbf{A}$ *is finitely related*

Example: Every semilattice is finitely related

Example: If $\mathrm{HSP}(\mathbf{C}) = \mathrm{HSP}(\mathbf{D})$ then $\mathbf{C}$ is finitely related iff $\mathbf{D}$ is finitely related

Partial reduction to the idempotent case:

Fact

Assume all unary term operations of $\mathbf{A}$ *are bijections. Then* $\mathbf{A}$ *is finitely related iff its full idempotent reduct is.*

Fininite relatedness and constructions - negative results

Fact (Davey, Jackson, Pitkethly, Szabó)

None of the constructions H,S,P preserve finite relatedness.

Their algebras were non-idempotent

Fininite relatedness and constructions - negative results

Fact (Davey, Jackson, Pitkethly, Szabó)

None of the constructions H,S,P preserve finite relatedness.

Their algebras were non-idempotent

Fact

Even in the idempotent case H,S does not preserve finite relatedness.

Fininite relatedness and constructions - negative results

Fact (Davey, Jackson, Pitkethly, Szabó)

None of the constructions H,S,P preserve finite relatedness.

Their algebras were non-idempotent

Fact

Even in the idempotent case H,S does not preserve finite relatedness.

Fact

The condition (*) is preserved by products.

Fininite relatedness and constructions - negative results

Fact (Davey, Jackson, Pitkethly, Szabó)

None of the constructions H,S,P preserve finite relatedness.

Their algebras were non-idempotent

Fact

Even in the idempotent case H,S does not preserve finite relatedness.

Fact

The condition (*) is preserved by products.

Problem

Find two finitely related idempotent algebras whose product is not finitely related.

A non-idempotent example

$\mathbf{A} = (\{0, 1, 2\}, *)$, where

$*$	0	1	2
0	0	0	0
1	0	0	2
0	0	1	2

A is not finitely related Davey, Jackson, Pitkethly, Szabó

A non-idempotent example

$\mathbf{A} = (\{0, 1, 2\}, *)$, where

$*$	0	1	2
0	0	0	0
1	0	0	2
0	0	1	2

A is not finitely related Davey, Jackson, Pitkethly, Szabó

Problem

Find some general explanation for this example.

A non-idempotent example

$\mathbf{A} = (\{0, 1, 2\}, *)$, where

$*$	0	1	2
0	0	0	0
1	0	0	2
0	0	1	2

A is not finitely related Davey, Jackson, Pitkethly, Szabó

Problem

Find some general explanation for this example.

This is an example of a graph algebra

A non-idempotent example

A $= (\{0,1,2\}, *)$, where

$*$	0	1	2
0	0	0	0
1	0	0	2
0	0	1	2

A is not finitely related Davey, Jackson, Pitkethly, Szabó

Problem

Find some general explanation for this example.

This is an example of a graph algebra

Fact

All idempotent graph algebras are finitely related.

A non-idempotent example

A $= (\{0, 1, 2\}, *)$, where

$*$	0	1	2
0	0	0	0
1	0	0	2
0	0	1	2

A is not finitely related Davey, Jackson, Pitkethly, Szabó

Problem

Find some general explanation for this example.

This is an example of a graph algebra

Fact

All idempotent graph algebras are finitely related.

Problem

Characterize finitely related graph algebras.

Summary for idempotent algebras

Summary for idempotent algebras

- Clones with a cube term are precisely the upward inherently finitely related clones

Summary for idempotent algebras

- Clones with a cube term are precisely the upward inherently finitely related clones
- The only known reason for non-finite relatedness is that $\exists B\ B \triangleleft_j \mathbf{A}$ and $B \not\triangleleft \mathbf{A}$

Summary for idempotent algebras

- Clones with a cube term are precisely the upward inherently finitely related clones
- The only known reason for non-finite relatedness is that $\exists B\ B \triangleleft_j \mathbf{A}$ and $B \not\triangleleft \mathbf{A}$
- Weird Guess: this is the only reason

Summary for idempotent algebras

- Clones with a cube term are precisely the upward inherently finitely related clones
- The only known reason for non-finite relatedness is that $\exists B \ B \lhd_j \mathbf{A}$ and $B \not\lhd \mathbf{A}$
- Weird Guess: this is the only reason
- You disprove it, if

Summary for idempotent algebras

- Clones with a cube term are precisely the upward inherently finitely related clones
- The only known reason for non-finite relatedness is that $\exists B \; B \triangleleft_j \mathbf{A}$ and $B \not\triangleleft \mathbf{A}$
- Weird Guess: this is the only reason
- You disprove it, if
  - you find CM algebras with no cube term and no Jónsson absorption

Summary for idempotent algebras

- Clones with a cube term are precisely the upward inherently finitely related clones
- The only known reason for non-finite relatedness is that $\exists B\ B \triangleleft_j \mathbf{A}$ and $B \not\triangleleft \mathbf{A}$
- Weird Guess: this is the only reason
- You disprove it, if
  - you find CM algebras with no cube term and no Jónsson absorption
  - you find two finitely related algebras whose product is not finitely related

Summary for idempotent algebras

- Clones with a cube term are precisely the upward inherently finitely related clones
- The only known reason for non-finite relatedness is that $\exists B \; B \triangleleft_j \mathbf{A}$ and $B \not\triangleleft \mathbf{A}$
- Weird Guess: this is the only reason
- You disprove it, if
  - you find CM algebras with no cube term and no Jónsson absorption
  - you find two finitely related algebras whose product is not finitely related
  - you find a long chain of clones alternately fin. rel. and non-fin. rel.

Summary for idempotent algebras

- Clones with a cube term are precisely the upward inherently finitely related clones
- The only known reason for non-finite relatedness is that $\exists B \; B \triangleleft_j \mathbf{A}$ and $B \not\triangleleft \mathbf{A}$
- Weird Guess: this is the only reason
- You disprove it, if
  - you find CM algebras with no cube term and no Jónsson absorption
  - you find two finitely related algebras whose product is not finitely related
  - you find a long chain of clones alternately fin. rel. and non-fin. rel.
  - you find a clone of operations compatible with all graphs of bijections which is not finitely related

Summary for idempotent algebras

- Clones with a cube term are precisely the upward inherently finitely related clones
- The only known reason for non-finite relatedness is that $\exists B\ B \triangleleft_j \mathbf{A}$ and $B \not\triangleleft \mathbf{A}$
- Weird Guess: this is the only reason
- You disprove it, if
  - you find CM algebras with no cube term and no Jónsson absorption
  - you find two finitely related algebras whose product is not finitely related
  - you find a long chain of clones alternately fin. rel. and non-fin. rel.
  - you find a clone of operations compatible with all graphs of bijections which is not finitely related
- **Open problem:** Characterize downward inherently finitely related clones.

Summary for non-idempotent algebras

Summary for non-idempotent algebras

- Some graph algebras, semigroups, and others, are not finitely related and this cannot be explained by Weird Guess

Summary for non-idempotent algebras

- Some graph algebras, semigroups, and others, are not finitely related and this cannot be explained by Weird Guess
- **Open problem:** Characterize finitely related graph algebras

Summary for non-idempotent algebras

- Some graph algebras, semigroups, and others, are not finitely related and this cannot be explained by Weird Guess
- **Open problem:** Characterize finitely related graph algebras
- **Open problem:** Find some general explanation(s) for known examples, make a Weirder Guess.

Summary for non-idempotent algebras

- Some graph algebras, semigroups, and others, are not finitely related and this cannot be explained by Weird Guess
- **Open problem:** Characterize finitely related graph algebras
- **Open problem:** Find some general explanation(s) for known examples, make a Weirder Guess.

Thank you!