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Slices of essentially algebraic categories

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Abstract This paper is a contribution to the theory of functor slices of J. Sichler and V. Trnková. For every ordinal α we introduce a basket $\mathbb{E}_\alpha$, prove that every essentially algebraic category of height α is a slice of $\mathbb{E}_\alpha$, characterize small slices of $\mathbb{E}_\alpha$ and give a common generalization of known results about slices of the algebraic basket $\mathbb{A}$.

Keywords functor slice, baskets of concrete categories, essentially algebraic category, closure rule

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1 Introduction

In [9], J. Sichler and V. Trnková introduced a concept of functor slices. Their theory yields a quasiorder (i.e. a reflexive and transitive relation) $\leq_s$ on the collection of all faithful functors and thus determines an equivalence $\sim_s$ by $U \sim_s V$ iff $U \leq_s V$ and $V \leq_s U$. If $U \leq_s V$, they say that U is a *slice* of V. See Section 3 for the corresponding definitions.

The results in [9] and more recent investigations [10], [7], [3] have shown an interesting and surprising phenomenon: Forgetful functors of many familiar concrete categories belong to one of five $\sim_s$ equivalence "classes", which were named *baskets*. These baskets together with $\leq_s$ inequalities between them are indicated in Figure 1 (an arrow stands for $\leq_s$; none of the arrows reverses and no arrow can be added, except the arrows implied by transitivity and reflexivity, of course).

Loosely speaking, the basket $\mathbb{R}$ contains the concrete categories (we mean their forgetful functors) which choose their morphisms "in a relational way"; those categories which choose their morphisms "algebraically" are in the basket $\mathbb{A}$; the baskets $\mathbb{P}, \mathbb{P}^{op}$

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$$
\begin{tikzcd}
& \mathbb{R} & \\
& \mathbb{A} \arrow[u] & \\
\mathbb{P} \arrow[ur] & & \mathbb{P}^{op} \arrow[ul] \\
& \mathbb{T} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Fig. 1 The five basic baskets

consist of "degenerate" cases of categories from $\mathbb{A}$; the trivial basket $\mathbb{T}$ contains precisely full embeddings.

However, as was observed later by J. Sichler and V. Trnková, there are many "natural" baskets which lie strictly between $\mathbb{A}$ and $\mathbb{R}$. For example, the category whose objects are sets with two unary operation, the first one total and the second one partial, defined precisely where the first operation has a fix-point. This category determines the basket $\mathbb{E}_2$. We can add a third unary operation defined on fix-points of the second one and we obtain the basket $\mathbb{E}_3$. Continuing in a similar fashion, we get a basket $\mathbb{E}_\alpha$ for every ordinal α. The slice ordering between $\mathbb{E}_\alpha$ and their duals is shown in Figure 2.

Fig. 2 Baskets of essentially algebraic categories

$$
\begin{tikzcd}
& \mathbb{R} & \\
\vdots \arrow[ur] & & \vdots \arrow[ul] \\
\mathbb{E}_\omega & & \mathbb{E}_\omega^{op} \\
\vdots & & \vdots \\
\mathbb{E}_2 \arrow[u] & & \mathbb{E}_2^{op} \arrow[u] \\
& \mathbb{A} = \mathbb{E}_1 = \mathbb{E}_1^{op} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

These categories are special cases of so called essentially algebraic categories (see [2], Section 4). Our first major theorem says that every essentially algebraic category is a slice of some $\mathbb{E}_\alpha$. An important example of an essentially algebraic category is the category of small categories. We show that it belongs to the basket $\mathbb{E}_2$.

The reason why no arrow in Figure 1 can be reversed or added is that certain properties of faithful functors are inherited to slices: Every slice of any member of $\mathbb{R}$ is SSF (strongly small fibered, [10], see Section 3), every slice of (any member of) $\mathbb{A}$ obeys Isbell's [4,5] zig-zag condition (zz) [9], every slice of $\mathbb{P}$ obeys (p), every slice of $\mathbb{P}^{op}$ obeys $(p)^{op}$ [9]. Conditions $(zz), (p), (p)^{op}$ are recalled in Section 5, we call them "closure rules". We introduce "multiple zig-zag closure rules" (zz^α) which are obeyed by all slices of $\mathbb{E}_\alpha$ and show that no arrow in Figure 2 can be reversed or added (except the obvious arrows, again).

On the other hand, these properties are known to be sufficient in the following cases: every SSF faithful functor is a slice of $\mathbb{R}$, every SSF faithful functor which obeys (p) (resp. (p^{op})) is a slice of $\mathbb{P}$ (resp. $\mathbb{P}^{op}$) (see [10]). Only partial results are known about the basket $\mathbb{A}$: If $U : \mathbf{K} \to \mathbf{H}$ is a faithful functor which obeys (zz) and either $\mathbf{K}$ and $\mathbf{H}$ are small [9], or U is SSF and $\mathbf{H} = \mathbf{Set}$ [8], then U is a slice of $\mathbb{A}$. We prove in Section 6 that every faithful functor between small categories which obeys (zz^α) is a slice of $\mathbb{E}_\alpha$. We also give a slight generalization of both above mentioned results about the basket $\mathbb{A}$.

The paper is organized as follows:

Section 2	Preliminaries and notation.
Section 3	The concept of a functor slice, equivalent formulations; the baskets $\mathbb{R}, \mathbb{A}, \mathbb{P}, \mathbb{P}^{op}, \mathbb{T}$; SSF condition.
Section 4	The definition of essentially algebraic category of height α; the baskets $\mathbb{E}_\alpha$; every essentially algebraic category of height α is a slice of $\mathbb{E}_\alpha$.
Section 5	Closure rule, obeying a closure rule, semantic consequence; the closure rules (zz^α); every essentially algebraic category of height α obeys (zz^α); no arrow in Figure 2 can be added or reversed; syntactic and semantic consequences of closure rules.
Section 6	Known results about universality with respect to closure rules; every faithful functor between small categories which obeys (zz^α) is a slice of $\mathbb{E}_\alpha$; slices of $\mathbb{A}$.

2 Preliminaries and notation

2.1 Category theory

To the basics we refer to [1].

The set of all morphisms in a category $\mathbf{K}$ with domain $A \in \mathrm{Obj}(\mathbf{K})$ and codomain $B \in \mathrm{Obj}(\mathbf{K})$ is denoted by $\mathbf{K}(A, B)$.

Given a faithful functor $U : \mathbf{K} \to \mathbf{H}$, $A, B \in \mathrm{Obj}(\mathbf{K})$ and $f \in \mathbf{H}(UA, UB)$ we say that f *carries* a $\mathbf{K}$-morphism from A to B provided that $f = Ug$ for a $\mathbf{K}$-morphism $g : A \to B$.

By a *concrete category* (over $\mathbf{H}$) we mean a faithful functor $U : \mathbf{K} \to \mathbf{H}$ such that

$$\mathbf{K}(A, B) \subseteq \mathbf{H}(UA, UB), \quad A, B \in \mathrm{Obj}(\mathbf{K}).$$

In this case, a $\mathbf{H}$-morphism $h : UA \to UB$ carries a $\mathbf{K}$-morphism $A \to B$ iff it *is* a $\mathbf{K}$-morphism $A \to B$.

We write $h \in \mathbf{H}(A, B)$, or h is a $\mathbf{H}$-morphism from A to B, in place of $h \in \mathbf{H}(UA, UB)$. Likewise, for $A \in \mathrm{Obj}(\mathbf{K})$, $H \in \mathrm{Obj}(\mathbf{H})$ we write $h \in \mathbf{H}(A, H)$ in place of $h \in \mathbf{H}(UA, H)$.

Let $\mathbf{H}$ be a category and $F, G : \mathbf{H} \to \mathbf{Set}$ be functors. The category $\mathbf{A}[F, G]$ is defined as follows: Objects are pairs (H, α), where $H \in \mathrm{Obj}(\mathbf{H})$ and $\alpha \in \mathbf{Set}(FH, GH)$. An $\mathbf{H}$-morphism $h : H \to H'$ is an $\mathbf{A}[F, G]$-morphism from (H, α) to (H', α'), if $Gh \circ \alpha = \alpha' \circ Fh$. We have a natural forgetful functor $\mathbf{A}[F, G] \to \mathbf{H}$ sending (H, α) to H.

2.2 Set theory

We work in a standard set theory with axiom of choice for classes. At several places we use "collections larger than classes" for the sake of brevity. This can be made correct by enhancing the set theory (see [1]), but, in this article, everything could be formulated without any use of such monsters.

An *ordinal* is a set of all smaller ordinals and cardinal is the least ordinal with its cardinality. We write $\alpha < \beta$ in place of $\alpha \in \beta$.

A partially ordered set $(P, <)$ (= poset) is said to be *well-founded* provided that every nonempty subset has a $<$-minimal element. The *rank* function from P to the class of ordinals is the unique function which satisfy (see [6])

$$\mathrm{rank}_P(p) = \begin{cases} 0 & \text{there is no } q < p, \\ \sup\{\mathrm{rank}_P(q) + 1 \mid q < p\} & \text{otherwise.} \end{cases}$$

By the *height* of P is meant the ordinal number $\sup\{\mathrm{rank}_P(p) + 1 \mid p \in P\}$, or 0 if P is empty. The subscripts will be omitted, if they are clear from the context. A *tree* is a well-founded poset such that the set $\{q \mid q < p\}$ is well-ordered for all $p \in P$.

The symbols $\sqcup, \coprod$ are used for the *coproduct* of sets, i.e. the disjoint union. Since, as I hope, there is no danger of confusion, we identify components of a coproduct with the sets from which the coproduct is formed, so that $A, B \subseteq A \sqcup B$, for instance.

2.3 Algebra

The notation here follows the monograph [2].

Let S be a set (of *sorts*). By an *S-sorted signature* is understood a set Σ of *operational symbols* together with an *arity* function assigning to every $\sigma \in \Sigma$ a κ-tuple $(s_i)_{i<\kappa}$ of sorts for some cardinal number κ and a sort s. Notation:

$$\sigma : \prod_{i<\kappa} s_i \to s.$$

A signature is called *nullary*, if it contains nullary operational symbols only. Otherwise, the signature is *nonnullary*.

By an S-sorted set is meant a family $(A_s)_{s\in S}$ of sets. A partial algebra $\mathcal{A}$ of the signature Σ is a pair $((A_s)_{s\in S}, (\sigma^{\mathcal{A}})_{\sigma\in\Sigma})$, where A_i are sets and $\sigma^{\mathcal{A}}$ are partial operations

$$\sigma^{\mathcal{A}} : \mathrm{Def}(\sigma^{\mathcal{A}}) \subseteq \prod_{i<\kappa} A_{s_i} \to A_s.$$

Operations with the *definition domain* $\mathrm{Def}(\sigma^{\mathcal{A}})$ equal to $\prod_{i<\kappa} A_{s_i}$ are called *total*.

A *homomorphism* from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ is a family of mappings $f = (f_s)_{s\in S}, f_s : A_s \to B_s$ preserving the operations in the following sense: If $\sigma : \prod_{i<\kappa} s_i \to s$ and $(a_i)_{i<\kappa} \in \mathrm{Def}(\sigma^{\mathcal{A}})$, then $(f(a_i))_{i<\kappa} \in \mathrm{Def}(\sigma^{\mathcal{B}})$ and

$$f_s(\sigma^{\mathcal{A}}(a_i)) = \sigma^{\mathcal{B}}(f(a_i)).$$

This yields the category $\mathbf{Palg}(\Sigma)$ of all partial algebras of the signature Σ and their homomorphisms, $\mathbf{Alg}(\Sigma)$ is its full subcategory formed by algebras with all operations total.

The set of *terms* (or Σ-terms) over an S-sorted set X of variables is the smallest S-sorted set such that

- each variable of sort s is a term of sort s,
- for each operational symbol $\sigma : \prod_{i<\kappa} s_i \to s$ and κ-tuple of terms τ_i of sort s_i, we conclude that $\sigma(\tau_i)$ is a term of sort s.

Given an algebra $\mathcal{A}$, term t and a family $(a_x)_{x\in X}$ of elements of the underlying S-sorted set of $\mathcal{A}$ we can naturally define the *value* $t^{\mathcal{A}}(a_x)$ of $t^{\mathcal{A}}$ in (a_x) for those (a_x) which are in the *definition domain* $\mathrm{Def}(t^{\mathcal{A}})$ of the term $t^{\mathcal{A}}$.

In this paragraph we assume that the signature Σ contains no nullary operational symbol. By an *address* we mean a finite (possible empty) sequence of ordinal numbers. The concatenation of addresses R, S is denoted by $R\hat{\ }S$. By a subterm of a term t at the address R, we mean the term $t[R]$ defined inductively by

1. $\tau[\emptyset] = \tau$.
2. If $R = S\hat{\ }i$, $\tau[S] = \sigma(\tau_i)_{i<\kappa}$ and $i < \kappa$, then $\tau[R] = \tau_i$; otherwise $\tau[R]$ is undefined.

If $\tau[R]$ is defined, we say that R is a *valid* address of τ. The valid addresses which have maximal length are addresses of *leaves*, i.e. variables in τ. The operational symbol at a valid address R of τ is denoted by $\tau\langle R\rangle$.

An *(S-)equation* is a pair (τ_1, τ_2) of terms over X of the same sort. Notation: $\tau_1 = \tau_2$. An equation $\tau_1 = \tau_2$ is *satisfied* by an algebra $\mathcal{A}$ in the elements $(a_x)_{x\in X}$ provided that $\tau_1^{\mathcal{A}}(a_x)$, $\tau_2^{\mathcal{A}}(a_x)$ are defined and equal. An algebra $\mathcal{A}$ satisfies $\tau_1 = \tau_2$ provided that $\tau_1^{\mathcal{A}}(a_x) = \tau_2^{\mathcal{A}}(a_x)$ whenever $(a_x)_{x\in X} \in \mathrm{Def}(t_1^{\mathcal{A}}), \mathrm{Def}(t_2^{\mathcal{A}})$.

3 Slices

The notion of a functor slice was introduced in [9]:

Definition 1 Let $U : \mathbf{K} \to \mathbf{H}$, $U' : \mathbf{K}' \to \mathbf{H}'$ be faithful functors. A pair (Φ, F) of functors $\Phi : \mathbf{K} \to \mathbf{K}'$, $F : \mathbf{H} \to \mathbf{H}'$ is said to be an *s-embedding* of U to U', if $FU = U'\Phi$ and for every $A, B \in \mathrm{Obj}(\mathbf{K})$, $f \in \mathbf{H}(A, B)$

$$\text{if } Ff \text{ carries a } \mathbf{K}'\text{-morphism } \Phi A \to \Phi B, \text{ then } f \text{ carries a } \mathbf{K}\text{-morphism } A \to B. \quad (1)$$

$$\begin{tikzcd} \mathbf{K} \arrow[r, "U"] \arrow[d, "\Phi"'] & \mathbf{H} \arrow[d, "F"] \\ \mathbf{K}' \arrow[r, "U'"'] & \mathbf{H}' \end{tikzcd} \quad (2)$$

If there exists an s-embedding of U to U', we say that U is a *slice* of U' and write $U \leq_s U'$. If $U \leq_s U'$ and $U' \leq_s U$, we say that U and U' are *s-equivalent* and write $U \sim_s U'$. The equivalence "classes" of $\sim_s$ are called *baskets*.

Remark 1 1. In the original definition from [9], the functor F (and thus the functor Φ) was assumed to be faithful. I think that the present definition is more workable and almost equally strong.
2. It is easy to see that $\leq_s$ is a quasiorder (reflexive and transitive) and thus $\sim_s$ is an equivalence relation. The notation $X \leq_s Y$ can (and will) be used, if X, Y are baskets, or if X is a faithful functor and Y is a basket, etc.
3. (Φ, Id) is an s-embedding iff Φ is concrete (that means $U'\Phi = U$), full and faithful.
4. If U, U' are concrete categories (see Preliminaries), what we can (and often will) assume, the condition (1) can be formulated as follows:
$$\text{If } Ff \in \mathbf{K}'(\Phi A, \Phi B) \text{ then } f \in \mathbf{K}(A, B). \tag{3}$$
5. An s-embedding is a weaker notion than a strong embedding: If (Φ, F) is an s-embedding and F is faithful, then (Φ, F) is a strong embedding iff every $\mathbf{K}'$-morphism $g : \Phi A \to \Phi B$ is of the form $g = Ff$ for some $\mathbf{H}$-morphism $f : UA \to UB$.

To avoid verbose statements, we will often say that "a category $\mathbf{K}$ is a slice of a category $\mathbf{H}$", in place of "the natural forgetful functor of $\mathbf{K}$ is a slice of the natural forgetful functor of $\mathbf{H}$", if the meaning of "natural" is clear.

A commutative diagram (2) such that (Φ, F) is an s-embedding is called a *subpullback* for the following reason (see [9]).

Proposition 1 *Let $U : \mathbf{K} \to \mathbf{H}$, $U' : \mathbf{K}' \to \mathbf{H}'$ be faithful functors and $(\Phi : \mathbf{K} \to \mathbf{K}', F : \mathbf{H} \to \mathbf{H}')$ be a pair of functors such that $FU = U'\Phi$. Then the following statements are equivalent.*

(i) *(Φ, F) is an s-embedding.*

(ii) *For every $A, B \in \mathrm{Obj}(\mathbf{K})$, the following diagram is a pullback in* $\mathbf{Set}$.

$$\begin{tikzcd}
\mathbf{K}(A,B) \arrow[r, "U"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{H}(UA,UB) \arrow[d, "F"] \\
\mathbf{K}'(\Phi A, \Phi B) \arrow[r, "U'"'] & \mathbf{H}'(U'\Phi A, U'\Phi B)
\end{tikzcd}$$

(iii) *The functor I in the following commutative diagram is a full embedding. (The mark at the top-left corner of the square denotes pullback.)*

$$\begin{tikzcd}
\mathbf{K} \arrow[drr, "U"] \arrow[dr, dashed, "I"] \arrow[ddr, "\Phi"'] & & \\
& {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{H} \arrow[d, "F"] \\
& \mathbf{K}' \arrow[r, "U'"'] & \mathbf{H}'
\end{tikzcd}$$

Corollary 1 *Let $U : \mathbf{K} \to \mathbf{H}$, $V : \mathbf{H} \to \mathbf{L}$ be faithful functors. Then $U \leq_s VU$.*

Proof It is easy to see that (Id, V) is an s-embedding. $\square$

Corollary 2 *Let $U : \mathbf{K} \to \mathbf{H}$, $U' : \mathbf{K}' \to \mathbf{H}'$ be faithful functors. Then $U \leq_s V$ iff $U^{op} : \mathbf{K}^{op} \to \mathbf{H}^{op} \leq_s V^{op} : \mathbf{K}'^{op} \to \mathbf{H}'^{op}$.*

Now, we mention some members of the baskets in Figure 1.

Basket $\mathbb{R}$ contains (see [9]) the category $\mathbf{Rel}(\Sigma)$ of relational structures and their homomorphisms for every nonnulary mono-sorted signature; the category $\mathbf{Palg}(\Sigma)$ for every nonnullary mono-sorted signature; the category **Pos** of all partially ordered sets (posets) and order preserving mappings; the category **Top** of all topological spaces and continuous mappings and all its full subcategories down to the category of all metrizable spaces; the category **Unif** of all uniform spaces and uniformly continuous mappings and all its full subcategories down to the category of all complete metrizable spaces; the category **Metr** of all metric spaces and maps which do not increase the distance and all its full subcategories down to the category of all complete metric spaces of diameter at most one; all their duals.

Basket $\mathbb{A}$ contains the category $\mathbf{Alg}(\Sigma)$ for every nonnullary mono-sorted signature (see [9]); more generally the category $\mathbf{Set}^T$ of all monadic algebras for any non-degenerate monad T over **Set** (see [7]; a monad is non-degenerate iff its functor part T is neither the identity nor a constant nor their coproduct); the category $\mathbf{Set}_T$ of all comonadic coalgebras for any non-degenerate comonad T over **Set** (see [3]); all their duals [9].

Basket $\mathbb{P}$ contains the category $\mathbf{Alg}(\Sigma)$ for a nullary nonempty mono-sorted signature [9].

Basket $\mathbb{P}^{op}$ contains precisely the duals of categories in $\mathbb{P}$ [9].

Basket $\mathbb{T}$ consists of all full and faithful functors [9].

An important property which is inherited to slices is the SSF condition (see [1]):

Definition 2 A concrete category $U : \mathbf{K} \to \mathbf{H}$ is said to be *SSF (strongly small fibered)*, if for every $H \in \mathrm{Obj}(\mathbf{H})$, the following equivalence $\sim_{SSF}$ on the class of all pairs (K, f), where $K \in \mathrm{Obj}(\mathbf{K})$, $f \in \mathbf{H}(K, H)$, has only set-many equivalence classes:

$$(K, f) \sim_{SSF} (K', f')$$

iff

$$(\forall L \in \mathrm{Obj}(\mathbf{K}))\ (\forall g \in \mathbf{H}(H, L))\quad gf \in \mathbf{K}(K, L) \Leftrightarrow gf' \in \mathbf{K}'(K', L)$$

Most of "everyday life" categories are SSF. All categories mentioned in this paper, for instance.

Proposition 2 (See [10]) *A slice of SSF concrete category is SSF.*

On the other hand, every SSF concrete category is a slice of $\mathbb{R}$. See Section 6 for this and similar results.

4 Essentially algebraic categories

As mentioned, the category $\mathbf{Palg}(\Sigma)$ of all partial algebras with given (nonnullary) signature and their homomorphisms belongs to the relational basket (we mentioned the mono-sorted case only, but this can be easily generalized). However, these categories have important full subcategories called essentially algebraic. These categories substantially enrich our five-member collection of baskets.

Definition 3 Let α be an ordinal, S be a set. An S-sorted *essentially algebraic theory* of height α is given by a quadruple $\Gamma = (\Sigma, \mathrm{level}, E, \mathrm{Def})$ where:

- Σ is an S-sorted signature (finitary or infinitary).
- $\mathrm{level} : \Sigma \to \alpha$ is a mapping assigning a *level* to each operational symbol $\sigma \in \Sigma$. The set of all operational symbols of level β is denoted by Σ_β. Analogically we define $\Sigma_{<\beta}$, $\Sigma_{\leq\beta}$.
- E is a set of Σ-equations.
- Def assigns to each κ-ary operational symbol $\sigma \in \Sigma$ a set of $\Sigma_{<\mathrm{level}(\sigma)}$-equations over a κ-indexed set $X = (x_i)_{i<\kappa}$ (where the variables have the right sorts). For all σ such that $\mathrm{level}(\sigma) = 0$, we assume $\mathrm{Def}(\sigma) = \emptyset$.

By a *model* of Γ (or a Γ-*algebra*) we mean a partial S-sorted algebra $\mathcal{A} = ((A_s)_{s\in S}, (\sigma^{\mathcal{A}})_{\sigma\in\Sigma})$ such that $\mathcal{A}$ satisfies all equations of E and $\sigma^{\mathcal{A}}(a_i)_{i<\kappa}$ is defined iff $\mathcal{A}$ satisfies all equations from $\mathrm{Def}(\sigma)$ in the elements $(a_i)_{i<\kappa}$.

The category of all Γ-algebras and homomorphisms is called an S-sorted *essentially algebraic category* of height α.

Remark 2 1. Locally presentable categories are, up to equivalence, precisely essentially algebraic categories (see [2]). In fact, essentially algebraic categories of height 2 suffice to describe all locally presentable categories at the abstract level (i.e. up to equivalence), but the height is significant at the concrete level (i.e. when considering forgetful functors).
2. Operations of level 0 are total. Operations of level 1 are defined where certain equations in total operational symbols are satisfied, and so on. This guarantees the following pleasant property of homomorphisms: Let ρ be a κ-ary operational symbol of level β. If a mapping $f : \mathcal{A} \to \mathcal{B}$ preserves all operations $\sigma \in \Sigma_{<\beta}$, then $(a_i)_{i<\kappa} \in \mathrm{Def}(\rho^{\mathcal{A}})$ implies $(f(a_i))_{i<\kappa} \in \mathrm{Def}(\rho^{\mathcal{B}})$.
3. An S-sorted essentially algebraic category of height 0 is (isomorphic to) the category $\mathbf{Set}^S$ of S-sorted sets.
4. S-sorted essentially algebraic categories of height 1 are precisely varieties of S-sorted algebras.
5. Let $\mathbf{K}$ be an S-sorted essentially algebraic category. We have two "natural" forgetful functors U, V: $U : \mathbf{K} \to \mathbf{Set}^S$ sends an algebra $\mathcal{A} = ((A_s)_{s\in S}, ...)$ to $(A_s)_{s\in S}$. $V : \mathbf{K} \to \mathbf{Set}$ sends $\mathcal{A}$ to $\coprod_{s\in S} A_s$.

For every well-founded poset P we now define a mono-sorted essentially algebraic category $\mathbf{Fix}(P)$ of height equal to the height of P. The important cases are $P = \alpha$ for an ordinal α with its natural ordering.

Definition 4 Let A be a set and M be a set of unary operations on A (possibly empty). The set of all common fix-points of all operations in M will be denoted by $\mathrm{Fix}(M)$:

$$\mathrm{Fix}(M) = \{a \in A \mid (\forall m \in M)\ m(a) = a\}.$$

Definition 5 Let $(P,<)$ be a well-founded poset. $\mathbf{Fix}(P)$ is the category of models of the essentially algebraic theory $\Gamma = (\Sigma, \text{level}, E, \text{Def})$, where Σ is mono-sorted and consists of unary operational symbols ϕ_p, $p \in P$; level(p) is the rank of p in the poset P; $E = \emptyset$; $\text{Def}(\phi_p) = \{\phi_q(x_0) = x_0 \mid q < p\}$.

So, an algebra $\mathcal{A} \in \text{Obj}(\mathbf{Fix}(P))$ is a set A together with partial unary operations $\phi_p^{\mathcal{A}}$, $p \in P$ such that $\text{Def}(\phi_p^{\mathcal{A}}) = \text{Fix}(\{\phi_q^{\mathcal{A}} \mid q < p\})$.

Let $\mathbb{E}_\alpha$ denote the basket determined by $\mathbf{Fix}(\alpha)$.

We will see that (any of the two forgetful functors of) each essentially algebraic category of height α is a slice of $\mathbb{E}_\alpha$ (Theorem 1) and we will characterize those functors between small categories which are slices of $\mathbb{E}_\alpha$ (Theorem 4).

We will show that the inequalities marked in Figure 2 hold and no arrow can be added or reversed: $\mathbb{E}_\alpha \leq_s \mathbb{E}_\beta$ for $\alpha \leq \beta$ (Proposition 3) and the inequality is strict if $\alpha < \beta$ (Proposition 7); $\mathbb{E}_2 \not\leq_s \mathbb{E}_\alpha^{op}$ for every α (Proposition 8); of course, $\mathbb{E}_\alpha \leq_s \mathbb{R}$, since every essentially algebraic category is a concrete full subcategory of $\mathbf{Palg}(\Sigma)$; $\mathbb{E}_\alpha \not\sim_s \mathbb{R}$ follows from Proposition 5, Proposition 6, Corollary 5.1., for instance.

Proposition 3 *Let P be a subposet of a poset Q. Then $\mathbf{Fix}(P) \leq_s \mathbf{Fix}(Q)$. In particular $\mathbb{E}_\alpha \leq_s \mathbb{E}_\beta$ for arbitrary ordinals $\alpha \leq \beta$.*

Proof Let $F = \text{Id}$. For an algebra $\mathcal{A} = (A, (\phi_p)_{p\in P}^{\mathcal{A}}) \in \mathbf{Fix}(P)$ let $\Phi\mathcal{A} = (A, (\phi_q)_{q\in Q}^{\Phi\mathcal{A}})$, where

$$\text{Def}(\phi_q^{\Phi\mathcal{A}}) = \text{Fix}(\{\phi_p^{\mathcal{A}} \mid p \in P, p < q\}),$$
$$\phi_q^{\Phi\mathcal{A}}(a) = \begin{cases} \phi_q^{\mathcal{A}}(a) & \text{if } q \in P, \\ a & \text{otherwise} \end{cases}$$

for all $a \in \text{Def}(\phi_q^{\Phi\mathcal{A}})$.

Clearly, $\Phi\mathcal{A}$ is a $\mathbf{Fix}(Q)$-object, Φ is a functor and (Φ, F) is an s-embedding. □

Theorem 1 *Let $\mathbf{K}$ be an S-sorted essentially algebraic category of height α with its theory $\Gamma = (\Sigma, \text{level}, E, \text{Def})$. Then $U \leq_s V \leq_s \mathbb{E}_\alpha$ where $U : \mathbf{K} \to \mathbf{Set}^S$, $V : \mathbf{K} \to \mathbf{Set}$ are the natural forgetful functors.*

Proof $U \leq_s V$ follows from Corollary 1 since V is the composition of U and the coproduct functor $\mathbf{Set}^S \to \mathbf{Set}$.

We can assume that $E = \emptyset$ (because concrete full subcategory is a slice of the original category) and that Σ contains no nullary operational symbol (we can replace them by unary operational symbols).

We can and will further assume that Σ is mono-sorted:

Claim V is a slice of a mono-sorted essentially algebraic category of height α.

Proof Let
$$\overline{\Gamma} = (\overline{\Sigma} = \Sigma \sqcup \{\rho\}, \overline{\text{level}}, \emptyset, \overline{\text{Def}}),$$

where operational symbols from $\Sigma \subseteq \overline{\Sigma}$ have the same arities, levels and defining identities, but are considered as mono-sorted (we forget sorts). The operational symbol ρ is unary and total (of level 0). The category of $\overline{\Gamma}$-algebras will be denoted by $\mathbf{L}$.

Now, we are going to define an s-embedding of V to (the natural forgetful functor of) $\mathbf{L}$. The functor F from the subpullback square (2) is defined by

$$\begin{aligned} FA &= A \sqcup S \sqcup \{c\}, \\ Ff &= f \sqcup \mathrm{id}_s \sqcup \mathrm{id}_c, \end{aligned}$$

where A is a set and $f : A \to B$ is a mapping.

The functor Φ is defined for an algebra $\mathcal{A} \in \mathbf{K}$ by

$$\Phi\mathcal{A} = \Phi((A_s)_{s\in S}, (\sigma^{\mathcal{A}})_{\sigma\in\Sigma}) = (\coprod_{s\in S} A_s \sqcup S \sqcup \{c\}, (\sigma^{\Phi\mathcal{A}})_{\sigma\in\Sigma}, \rho^{\Phi\mathcal{A}}),$$

where $\rho^{\Phi\mathcal{A}}(a_s) = s$ for $a_s \in A_s$, $\rho^{\Phi\mathcal{A}}(s) = \rho^{\Phi\mathcal{A}}(c) = c$ for $s \in S$. For an operational symbol $\sigma : \prod_{i<\kappa} s_i \to s$, the operation $\sigma^{\Phi\mathcal{A}} : \prod_{i<\kappa} FV\mathcal{A} \to FV\mathcal{A}$ is given by

$$\sigma^{\Phi\mathcal{A}}(a_i)_{i<\kappa} = \begin{cases} \sigma^{\mathcal{A}}(a_i)_{i<\kappa} & \text{if } a_i \in A_{s_i}, i < \kappa \text{ and } (a_i)_{i<\kappa} \in \mathrm{Def}(\sigma^{\mathcal{A}}), \\ c & \text{otherwise (on the def. dom.).} \end{cases}$$

It is easy to see that $\Phi\mathcal{A} \in \mathbf{L}$ for any $\mathcal{A} \in \mathbf{K}$.
Let $\mathcal{A} = ((A_s)_{s\in S}, \ldots), \mathcal{B} = ((B_s)_{s\in S}, \ldots) \in \mathbf{K}$. A mapping $f : \coprod_{s\in S} A_s \to \coprod_{s\in S} B_s$ carries a $\mathbf{K}$-homomorphism $\mathcal{A} \to \mathcal{B}$, iff $f(A_s) \subseteq B_s$ (for all $s \in S$) and f preserves all operations $\sigma \in \Sigma$. This arises precisely when $Ff : \Phi\mathcal{A} \to \Phi\mathcal{B}$ preserves ρ and all $\sigma \in \Sigma$. Hence (Φ, F) is an s-embedding. $\square$

To formulate and prove the next two claims which form the most technical part of this paper, we need to introduce further notation.

For a set X, let $Q_X : \mathbf{Set} \to \mathbf{Set}$ be the covariant hom-functor:

$$\begin{aligned} Q_X A &= \{(a_x)_{x\in X} \mid a_x \in A\}, && \text{where } A \text{ is a set,} \\ Q_X f(a_x)_{x\in X} &= (f(a_x))_{x\in X}, && \text{where } f : A \to B \text{ is a mapping.} \end{aligned}$$

Given a set Y a subset $D \subseteq Q_Y A$ and a set $X \subseteq Y$ we define a set $\mathrm{Proj}(D; Y \to X) \subseteq Q_X A$ by

$$\mathrm{Proj}(D; Y \to X) = \{(a_x)_{x\in X} \mid (\exists\ (b_y)_{y\in Y} \in D)\ (\forall x \in X)\ a_x = b_x\}.$$

Given a partial unary operation $\rho : \mathrm{Def}(\rho) \subseteq Q_X A \to Q_X A$ we define a partial unary operation $\mathrm{Ext}(\rho; X \to Y) : D \subseteq Q_Y A \to Q_Y A$ by

$$\begin{aligned} (a_y)_{y\in Y} \in D \ &\text{ iff } \ (a_x)_{x\in X} \in \mathrm{Def}(\rho), \\ (\mathrm{Ext}(\rho; X \to Y)(a_y)_{y\in Y})_k \ &= \ \begin{cases} (\rho(a_x)_{x\in X})_k & \text{if } k \in X, \\ a_k & \text{otherwise.} \end{cases} \end{aligned}$$

Given a subset $D \subseteq Q_X A$, an element $r \in X$ and a partial mapping $e : D \to A$ we define a partial unary operation $\mathrm{Ope}(D; e(a_x)_{x\in X} \to a_r)$ by

$$\begin{aligned} \mathrm{Def}(\mathrm{Ope}(D; e(a_x)_{x\in X} \to a_r)) &= D, \\ (\mathrm{Ope}(D; e(a_x)_{x\in X} \to a_r)(a_x)_{x\in X})_k &= \begin{cases} e(a_x)_{x\in X} & \text{if } k = r, \\ a_k & \text{otherwise.} \end{cases} \end{aligned}$$

Let P be a poset. We say that P satisfy (P1), if

(P1) *P is well-founded and the dual poset is a tree.*

The following two claims will be proved simultaneously by induction on β.

Claim (*) Let $\beta \leq \alpha$ be an ordinal. Let τ be a term over X in operational symbols from $\Sigma_{<\beta}$. Then there exists a poset P_τ of height $\leq \beta$ satisfying (P1), a set Y_τ and a functor $\Phi_\tau : \mathbf{K} \to \mathbf{Fix}(P_\tau)$ such that

(A1) $W\Phi_\tau = Q_{X \sqcup Y_\tau} V$, where $W : \mathbf{Fix}(P_\tau) \to \mathbf{Set}$ is the forgetful functor.
(A2) There is an element $z_\tau \in Y_\tau$ such that for each algebra $\mathcal{A} \in \mathbf{K}$

$$\mathrm{Proj}(\mathrm{Fix}(\{\phi_p^{\Phi_\tau \mathcal{A}} \,|\, p \in P_\tau\}); X \sqcup Y_\tau \to X \sqcup \{z_\tau\}) =$$

$$= \{(a_j)_{j \in X \sqcup \{z_\tau\}} \,|\, (a_x)_{x \in X} \in \mathrm{Def}(\tau^{\mathcal{A}}),\ a_{z_\tau} = \tau^{\mathcal{A}}(a_x)_{x \in X}\}.$$

(A3) Let $\mathcal{A} = (A, \ldots), \mathcal{B} = (B, \ldots) \in \mathbf{K}$. Let $f : A \to B$ be a mapping such that $Q_{X \sqcup Y_\tau} f : \Phi_\tau \mathcal{A} \to \Phi_\tau \mathcal{B}$ is a $\mathbf{Fix}(P_\tau)$-morphism.
Then $f(\tau^{\mathcal{A}}(a_x)_{x \in X}) = \tau^{\mathcal{B}}(f(a_x))_{x \in X}$ for any $(a_x)_{x \in X} \in \mathrm{Def}(\tau^{\mathcal{A}})$.

Claim (**) Let $\beta < \alpha$ be an ordinal, $\sigma \in \Sigma_{\leq \beta}$ be an operational symbol of arity κ, $X = (x_i)_{i < \kappa}$ be a κ-indexed set. Then there exists a poset P_σ of height $\leq \beta$ satisfying (P1), a set Y_σ and a functor $\Phi_\sigma : \mathbf{K} \to \mathbf{Fix}(P_\sigma)$ such that

(B1) $W\Phi_\sigma = Q_{X \sqcup Y_\sigma} V$, where $W : \mathbf{Fix}(P_\sigma) \to \mathbf{Set}$ is the forgetful functor.
(B2) For each algebra $\mathcal{A} \in \mathbf{K}$ we have

$$\mathrm{Proj}(\mathrm{Fix}(\{\phi_p^{\Phi_\sigma \mathcal{A}} \,|\, p \in P_\sigma\}); X \sqcup Y_\sigma \to X) = \{(a_x)_{x \in X} \,|\, (a_{x_i})_{i < \kappa} \in \mathrm{Def}(\sigma^{\mathcal{A}})\}.$$

Proof (of Claim (*)) Since the statement is empty for $\beta = 0$, we assume $\beta \geq 1$. Assume that Claim (**) holds for all $\gamma < \beta$. We denote

$$\begin{aligned}
\mathrm{Leaves} &= \{R \,|\, R \text{ is an address of a leaf of } \tau\}, \\
\mathrm{Addr} &= \{R \,|\, R \text{ is a valid address of } \tau,\ R \notin \mathrm{Leaves}\}, \\
\mathrm{Succ}(R) &= \{i \,|\, R\hat{\ }i \text{ is a valid address of } \tau\}, \qquad R \in \mathrm{Addr}, \\
Z_R &= \{z_{R\hat{\ }i} \,|\, i \in \mathrm{Succ(R)}\}, \qquad \mathrm{R} \in \mathrm{Addr}.
\end{aligned}$$

For all $R \in \mathrm{Addr}$ let Y_R be a set, P_R be a poset satisfying (P1) and $\Phi_R : \mathbf{K} \to \mathbf{Fix}(P_R)$ be a functor such that

- $W_R \Phi_R = Q_{Z_R \sqcup Y_R} V$, where $W_R : \mathbf{Fix}(P_R) \to \mathbf{Set}$ is the forgetful functor.
- For each algebra $\mathcal{A} \in \mathbf{K}$

$$\mathrm{Proj}(\mathrm{Fix}(\{\phi_p^{\Phi_R \mathcal{A}} \,|\, p \in P_R\}); Z_R \sqcup Y_R \to Z_R) =$$

$$= \{(a_z)_{z \in Z_R} \,|\, (a_{z_{R\hat{\ }i}})_{i < \kappa} \in \mathrm{Def}(t\langle R \rangle^{\mathcal{A}})\}.$$

Let

$$P_\tau = \coprod_{R \in \mathrm{Addr}} P_R \sqcup \{q_R \,|\, R \in \mathrm{Addr} \cup \mathrm{Leaves}\},$$

where the ordering of P_τ on the set P_R coincides with the ordering of P_R, q_R is a new greatest element of P_R for $R \in \mathrm{Addr}$ and q_R is of rank 0 for $R \in \mathrm{Leaves}$. The poset P_τ clearly satisfy (P1) and its height is not greater than β.

Let

$$Z = \{z_R \mid R \in \mathrm{Addr} \cup \mathrm{Leaves}\},$$
$$Y_\tau = \coprod_{R\in\mathrm{Addr}} Y_R \sqcup Z =$$
$$= \coprod_{R\in\mathrm{Addr}} Y_R \sqcup \coprod_{R\in\mathrm{Addr}} Z_R \sqcup \{z_\emptyset\}.$$

Finally we have to define the functor Φ_τ. For an algebra $\mathcal{A} = (A, (\sigma^{\mathcal{A}})_{\sigma\in\Sigma}) \in \mathbf{K}$ we put

$$\Phi_\tau\mathcal{A} = (Q_{X\sqcup Y_\tau}A, (\phi_p^{\Phi_\tau\mathcal{A}})_{p\in P_\tau}),$$

where

$$\phi_p^{\Phi_\tau\mathcal{A}} = \mathrm{Ext}(\phi_p^{\Phi_R\mathcal{A}}; Z_R \sqcup Y_R \to X \sqcup Y_\tau), \quad p \in P_R,$$
$$\phi_{q_R}^{\Phi_\tau\mathcal{A}} = \mathrm{Ope}(\mathrm{Fix}(\{\phi_p^{\Phi_\tau\mathcal{A}} \mid p \in P_R\}); \tau\langle R\rangle^{\mathcal{A}}(a_{z_{R^\frown i}})_{i\in\mathrm{Succ}(R)} \to a_{z_R}), \quad R \in \mathrm{Addr},$$
$$\phi_{q_R}^{\Phi_\tau\mathcal{A}} = \mathrm{Ope}(Q_{X\sqcup Y_\tau}A; a_{\tau\langle R\rangle} \to a_{z_R}), \quad R \in \mathrm{Leaves}.$$

From the properties of Φ_R we know that the definition of $\phi_{q_R}^{\Phi_\tau\mathcal{A}}$ makes sense. Clearly, if $f : \mathcal{A} \to \mathcal{B}$ is a homomorphism, then $Q_{X\sqcup Y_\tau}f : \Phi_\tau\mathcal{A} \to \Phi_\tau\mathcal{B}$ preserves the operation ϕ_p for all $p \in P_\tau$. Thus Φ_τ is a functor.

For $R \in \mathrm{Leaves}$ we have

$$\mathrm{Proj}(\mathrm{Fix}(\{\phi_{q_R}^{\Phi_\tau\mathcal{A}}\}); X \sqcup Y_\tau \to X \sqcup Z) = \{(a_j)_{j\in X\sqcup Z} \mid a_{z_R} = a_{\tau\langle R\rangle}\}$$

and for $R \in \mathrm{Addr}$ we have

$$\mathrm{Proj}(\mathrm{Fix}(\{\phi_{q_R}^{\Phi_\tau\mathcal{A}}\}); X \sqcup Y_\tau \to X \sqcup Z) =$$
$$= \{(a_j)_{j\in X\sqcup Z} \mid (a_{z_{R^\frown i}})_{i\in\mathrm{Succ}(R)} \in \mathrm{Def}(\tau\langle R\rangle^{\mathcal{A}}) \text{ and } a_{z_R} = \tau\langle R\rangle^{\mathcal{A}}(a_{z_{R^\frown i}})_{i\in\mathrm{Succ}(R)}\}.$$

Therefore

$$\mathrm{Proj}(\mathrm{Fix}(\{\phi_p^{\Phi_\tau\mathcal{A}} \mid p \in P_\tau\}); X \sqcup Y_\tau \to X \sqcup Z) =$$
$$= \{(a_j)_{j\in X\sqcup Z} \mid (\forall R \in \mathrm{Leaves} \cup \mathrm{Addr})\ (a_x)_{x\in X} \in \mathrm{Def}(\tau[R]^{\mathcal{A}}),\ a_{z_R} = \tau[R]^{\mathcal{A}}(a_x)_{x\in X}\}$$

and thus the property (A2) is satisfied for $z_\tau = z_\emptyset$ and (A3) is clear. $\square$

Proof (of (**)) The statement is clear for $\beta = 0$, thus we can assume $\beta \geq 1$. Assume that Claim (*) holds for all $\gamma \leq \beta$. Let $\mathrm{Def}(\sigma)$ consist of equations $\tau_i = \xi_i$, $i \in \lambda$, where τ and ξ are $\Sigma_{<\beta}$-terms over X. Let $Y_{\tau_i}, Y_{\xi_i}, z_{\tau_i}, z_{\xi_i}, P_{\tau_i}, P_{\xi_i}, \Phi_{\tau_i}, \Phi_{\xi_i}$ be from the induction hypothesis.

Let

$$Y_\sigma = (\coprod_{i<\lambda} Y_{\tau_i} \sqcup \coprod_{i<\lambda} Y_{\xi_i})/\approx$$
$$P_\sigma = \coprod_{i<\lambda} P_{\tau_i} \sqcup \coprod_{i<\lambda} P_{\xi_i}$$

where the ordering of P_σ on the sets P_{τ_i} and P_{ξ_i} coincides with the original one and no other inequalities are added; the equivalence $\approx$ glues z_{τ_i} with z_{ξ_i} and nothing else. The element $[z_{\tau_i}] = [z_{\xi_i}]$ of Y_σ will be denoted by z_i.

Now we define the functor Φ_σ. For an algebra $\mathcal{A} = (A, (\sigma^{\mathcal{A}})_{\sigma\in\Sigma}) \in \mathbf{K}$ we put

$$\Phi_\sigma\mathcal{A} = \{Q_{X\sqcup Y_\sigma}A, (\phi_p^{\Phi_\sigma\mathcal{A}})_{p\in P_\sigma}\},$$

where

$$\begin{aligned}\phi_p^{\Phi_\sigma\mathcal{A}} &= \mathrm{Ext}(\phi_p^{\Phi_{\tau_i}\mathcal{A}}; X\sqcup Y_{\tau_i} \to X\sqcup Y_\sigma), \quad p\in P_{\tau_i},\\ \phi_p^{\Phi_\sigma\mathcal{A}} &= \mathrm{Ext}(\phi_p^{\Phi_{\xi_i}\mathcal{A}}; X\sqcup Y_{\xi_i} \to X\sqcup Y_\sigma), \quad p\in P_{\xi_i}.\end{aligned}$$

Evidently, Φ_σ is a functor.

We have

$$(a_j)_{j\in X\sqcup\{z_i\,|\,i<\lambda\}} \in \mathrm{Proj}(\mathrm{Fix}(\{\phi_p^{\Phi_\sigma\mathcal{A}}\,|\,p\in P_\sigma\}); X\sqcup Y_\sigma \to X\sqcup\{z_i\,|\,i<\lambda\})$$

iff

$$(\forall i<\lambda)\ (a_x)_{x\in X}\in \mathrm{Def}(\tau_i^{\mathcal{A}})\cap\mathrm{Def}(\xi_i^{\mathcal{A}}) \text{ and } a_{z_i} = \tau_i^{\mathcal{A}}(a_x)_{x\in X} = \xi_i^{\mathcal{A}}(a_x)_{x\in X}$$

and (B2) follows. □

From Claim (*) we can now easily deduce:

Claim $\mathbf{K} \leq_s \mathbf{Fix}(P)$ for a poset P of height $\leq \alpha$ satisfying (P1).

Proof For every operational symbol $\sigma\in\Sigma$ we can use Claim (*) for the term $\sigma(x_i^\sigma)_{i\in\mathrm{arity}(\sigma)}$ over $X_\sigma = \{x_i^\sigma\}_{i\in\mathrm{arity}(\sigma)}$. We obtain a set Y_σ a poset P_σ of height at most α satisfying (P1) and a functor $\Phi_\sigma : \mathbf{K}\to\mathbf{Fix}(P_\sigma)$ such that

- $W_\sigma\Phi_\sigma = Q_{X_\sigma\sqcup Y_\sigma}V$,
- A mapping $f:\mathcal{A}\to\mathcal{B}$ preserves the operation σ whenever $Q_{X_\sigma\sqcup Y_\sigma} : \Phi_\sigma\mathcal{A}\to\Phi_\sigma\mathcal{B}$ is a $\mathbf{Fix}(P_\sigma)$-morphism.

Let

$$P = \coprod_{\sigma\in\Sigma} P_\sigma, \quad F = \coprod_{\sigma\in\Sigma} Q_{X_\sigma\sqcup Y_\sigma},$$

where the ordering of P on each component P_σ coincides with the original one and no other inequalities are added. Recall that the coproduct of functors is computed componentwise.

For an algebra $\mathcal{A} = (A,\dots)\in\mathbf{K}$, let $\Phi\mathcal{A} = (FA, (\phi_p^{\Phi\mathcal{A}})_{p\in P})$, where the operation $\phi_p^{\Phi\mathcal{A}}$ agrees with $\phi_p^{\Phi_\sigma\mathcal{A}}$ on the component $Q_{X_\sigma\sqcup Y_\sigma}A$ and $\phi_p^{\Phi\mathcal{A}}(x) = x$ for every $p\in P_\sigma$, $x\in FA - Q_{X_\sigma\sqcup Y_\sigma}A$. It is clear that Φ is a correctly defined functor and (Φ, F) is an s-embedding. □

To finish the proof we first adjust properties of the poset P and then find an s-embedding to $\mathbf{Fix}(\alpha)$. The wanted properties are:

(P2) *P is well-founded and $\{q\,|\,q>p\}$ is linearly (and hence well) ordered for every $p\in P$.*

(P3) *For every $p\in P$ and every ordinal β such that $\mathrm{rank}(p)<\beta<\alpha$, there exists a (unique) $q\in P$ for which $p<q$, $\mathrm{rank}(q)=\beta$.*

(P4) For every $p, p', q \in P$ such that $p, p' < q$ and $\mathrm{rank}(q)$ is a limit ordinal, there exists $r \in P$ such that $p, p' < r < q$;

Claim Every poset P of height $\leq \alpha$ satisfying (P1) is a subposet of some poset Q of height α which satisfy (P2), (P3) and (P4).

Proof Let $\overline{P}$ be the poset P with a new greatest element ∞:

$$\overline{P} = P \sqcup \{\infty\}, \quad p < \infty, \ p \in P.$$

Since the dual of $\overline{P}$ is a tree, we know that the interval $\langle p, p') = \{p'' \,|\, p \leq p'' < p'\}$ has a unique maximal element (for arbitrary $p, p' \in \overline{P}$, $p < p'$). Let

$$Q = P \sqcup \coprod_{p \in \overline{P}} Q_p,$$

where

$$\begin{aligned} Q_p &= \{q_{p,\beta} \,|\, 0 \leq \beta < \mathrm{rank}(p) \text{ is an ordinal } \}, \quad p \in P, \\ Q_\infty &= \{q_{\infty,\beta} \,|\, 0 \leq \beta < \alpha \text{ is an ordinal } \}. \end{aligned}$$

The ordering $<_Q$ is given by

$$\begin{array}{llll} p <_Q p' & \text{iff} \quad p <_P p' & \text{where} & p, p' \in P \\ q_{p,\beta} <_Q p' & \text{iff} \quad p \leq_P p' & \text{where} & p \in \overline{P}, p' \in P \\ p <_Q q_{p',\beta} & \text{iff} \quad p <_P p', & \text{where} & p \in P, \, p' \in \overline{P}, \\ & \qquad \beta > \mathrm{rank}_{\overline{P}}(\max_{\overline{P}}\langle p, p')) & & q_{p',\beta} \in Q_{p'} \\ q_{p,\beta} <_Q q_{p',\beta'} & \text{iff} \quad p = p', \, \beta < \beta' \text{ or} & \text{where} & p, p' \in \overline{P} \\ & \qquad p <_{\overline{P}} p', \, p <_Q q_{p',\beta'} & & q_{p,\beta} \in Q_p, \, q_{p',\beta'} \in Q_{p'} \end{array}$$

It is straightforward to verify that

- $<_Q$ is a partial ordering on Q.
- The function rank_Q given by $\mathrm{rank}_Q(p) = \mathrm{rank}_P(p)$ for $p \in P$ and $\mathrm{rank}_Q(q_{p,\beta}) = \beta$ for $p \in \overline{P}$, $q_{p,\beta} \in Q_p$ is the rank function of the poset Q. Hence Q is well-founded.
- If $q \in Q$ and β is an ordinal such that $\alpha > \beta > \mathrm{rank}(q)$, then there exists a unique $q' \in Q$ of Q-rank β such that $q <_Q q'$. Thus the properties (P2), (P3) are satisfied.
- Q satisfy (P4). This follows easily from the following fact: If $p, p', r \in P$, $p, p' <_P r$ and β is an ordinal such that $\mathrm{rank}_P(r) > \beta > \mathrm{rank}_P(\max\langle p, r))$, $\mathrm{rank}_P(\max\langle p', r))$, then $p, p' <_Q q_{r,\beta}$.

□

From the last claim and Proposition 3 we get $\mathbf{Fix}(P) \leq_s \mathbf{Fix}(Q)$.
Now it suffices to prove:

Claim Let P be a poset of height α satisfying (P2), (P3) and (P4). Then $\mathbf{Fix}(P) \leq_s \mathbf{Fix}(\alpha)$.

Proof For $\beta < \alpha$ let $P_\beta = \{p \in P \mid \mathrm{rank}(p) = \beta\}$ and for every $\beta < \gamma < \alpha$ let $s_{\beta,\gamma} : P_\beta \to P_\gamma$ be the mapping satisfying $p < s_{\beta,\gamma}(p)$, $p \in P_\beta$. Since P satisfy (P2) and (P3), $s_{\beta,\gamma}$ is a correctly defined surjective mapping and $p < q$ iff $s_{\mathrm{rank}(p),\mathrm{rank}(q)}(p) = q$.

Let $g : P_0 \to A$ be a mapping, $0 \leq \beta < \alpha$. If g factorizes through $s_{0,\beta}$, i.e. $g = g_\beta s_{0,\beta}$ for a mapping $g_\beta : P_\beta \to A$, we say that g_β exists. Since $s_{0,\beta}$ is surjective, if g_β exists then it is necessarily unique. From (P4) it follows that, for a limit β, g_β exists iff g_γ exists for all $\gamma < \beta$.

For sets A, B and a mapping $f : A \to B$ let

$$FA = \{(g,\beta) \mid g : P_0 \to A,\ \beta \leq \alpha\}/\approx,$$
$$Ff[g,\beta]_\approx = [fg,\beta]_\approx.$$

The equivalence $\approx$ is given by

$$(g,\beta) \approx (h,\gamma) \text{ iff both } g_{\max(\beta,\gamma)}, h_{\max(\beta,\gamma)} \text{ exist and } g = h.$$

F is clearly correctly defined and $\approx$ is an equivalence. We write $[\dots]$ instead of $[\dots]_\approx$.

Given $\mathcal{A} = (A, (\phi_p^{\mathcal{A}})_{p\in P}) \in \mathbf{Fix}(P)$, let

$$\Phi\mathcal{A} = (FA, (\phi_\beta^{\Phi\mathcal{A}})_{\beta<\alpha}),$$

where

$$\phi_0^{\Phi\mathcal{A}}[g,\beta] = [\overline{g}, 1], \quad \overline{g}(p) = \phi_p^{\mathcal{A}}(g(p)), \quad p \in P_0$$

and for $0 < \beta < \alpha$

$$\mathrm{Def}(\phi_\beta^{\Phi\mathcal{A}}) = \{[g,\beta] \mid g_\beta \text{ exists, } (\forall p \in P_\beta)\ g_\beta(p) \in \mathrm{Def}(\phi_p^{\mathcal{A}})\},$$
$$\phi_\beta^{\Phi\mathcal{A}}[g,\beta] = [\overline{g}, \beta^+], \quad \overline{g}(p) = \phi_{s_{0,\beta}(p)}^{\mathcal{A}}(g(p)), \quad p \in P_0.$$

To verify that $\Phi\mathcal{A}$ is a $\mathbf{Fix}(\alpha)$-object, we must check the following: For every $0 < \beta < \alpha$ we have $\mathrm{Fix}(\{\phi_\gamma^{\Phi\mathcal{A}} \mid \gamma < \beta\}) = \mathrm{Def}(\phi_\beta^{\Phi\mathcal{A}})$. By induction on β:

First step, $\beta = 1$: The element $[g,\beta] \in FA$ is a fix-point of $\phi_0^{\Phi\mathcal{A}}$ iff $[g,\beta] = [\overline{g}, 1]$, i.e. iff g_1 exists (which means that $g(p) = g(q)$ whenever $s_{0,1}(p) = s_{0,1}(q)$, where $p, q \in P_0$) and $\overline{g}(p) = \phi_p^{\mathcal{A}}(g(p)) = g(p)$ for all $p \in P_0$. This happens precisely when g_1 exists and $g_1(p) \in \mathrm{Fix}(\{\phi_q^{\mathcal{A}} \mid q \in s_{0,1}^{-1}(p)\}) = \mathrm{Def}(\phi_p^{\mathcal{A}})$ for all $p \in P_1$.

Isolated step is similar to the first step, limit step follows from the observation above: For a limit β, g_β exists iff g_γ exists for all $\gamma < \beta$.

Let f be a mapping $\mathcal{A} = (A, \dots) \to \mathcal{B} = (B, \dots)$. The mapping Ff preserves ϕ_0, iff for all $[g,\beta] \in FA$

$$\phi_0^{\Phi\mathcal{B}}(Ff[g,\beta]) = \phi_0^{\Phi\mathcal{B}}[fg,\beta] = [\overline{fg}, 1] =$$
$$= Ff(\phi_0^{\Phi\mathcal{A}}[g,\beta]) = Ff[\overline{g}, 1] = [f\overline{g}, 1].$$

For all $p \in P_0$

$$\overline{fg}(p) = \phi_p^{\mathcal{B}}(f(g(p)))$$

and

$$f(\overline{g}(p)) = f(\phi_p^{\mathcal{A}}(g(p))).$$

This means that Ff preserves ϕ_0, iff f preserves ϕ_p for all $p \in P_0$. Similarly, Ff preserves ϕ_β, iff f preserves $\phi_{s_{0,\beta}(p)}$ for all $p \in P_0$, i.e. iff f preserves ϕ_q for all $q \in P_\beta$. We can now see that Φ is a functor and (Φ, F) is an s-embedding. □

The proof of Theorem 1 is concluded. □

Problem 1 Find all baskets of essentially algebraic categories.

Remark 3 As mentioned, every mono-sorted essentially algebraic category of height 1 (i.e. a variety) belongs to one of the baskets $\mathbb{T}, \mathbb{P}, \mathbb{A}$. So that the first step could be to generalize this result to many-sorted signatures and then to look at (mono-sorted) essentially algebraic categories of height 2.

A natural example of an essentially algebraic category of height 2 is the category **Cat** of all small categories and functors (the forgetful functor **Cat** → **Set** assigns the set of all morphisms to a category). Indeed, **Cat** can be described as (i.e., is concretely equivalent to) the category of models of $\Gamma = (\{\circ, d, c\}, \mathrm{level}, E, \mathrm{Def})$, where

$$\mathrm{level}(d) = \mathrm{level}(c) = 0, \quad \mathrm{level}(\circ) = 1$$

are the operations of domain, codomain and comoposition, respectively.

$$\begin{aligned} E = \{&dd(x) = cd(x) = dm(x),\ cc(x) = dc(x) = cm(x),\\ &d(x \circ y) = d(y),\ c(x \circ y) = c(x),\\ &c(x) \circ x = x = x \circ d(x),\\ &x \circ (y \circ z) = (x \circ y) \circ z\},\\ \mathrm{Def}(\circ) = \{&d(x_0) = c(x_1)\}. \end{aligned}$$

This is just an object free definition of a category.

Proposition 4 *The category* **Cat** *is a member of* $\mathbb{E}_2$.

Proof Since $\mathbf{Cat} \leq_s \mathbf{Fix}(2)$ follows from Theorem 1, it suffices to find an s-embedding (Φ, F) of $\mathbf{Fix}(2)$ to **Cat**.

The functor $F : \mathbf{Set} \to \mathbf{Set}$ is defined by

$$\begin{aligned} FA &= \{m_{a,b},\ \mathrm{id}_{a,b,i} \mid a, b \in A,\ i \in 2\}/\approx,\\ Ff[m_{a,b}] &= [m_{f(a),f(b)}],\\ Ff[\mathrm{id}_{a,b,i}] &= [\mathrm{id}_{f(a),f(b),i}], \end{aligned}$$

where A is a set, $f : A \to B$ is a mapping, the equivalence $\approx$ is generated by $\mathrm{id}_{a,a,0} \approx \mathrm{id}_{a,a,1}$ for all $a \in A$, and $[\ldots]$ means $[\ldots]_\approx$.

For an algebra $\mathcal{A} = (A, (\phi_i^{\mathcal{A}})_{i\in 2}) \in \mathbf{Fix}(2)$ we put

$$\Phi\mathcal{A} = (FA, d^{\Phi\mathcal{A}}, c^{\Phi\mathcal{A}}, \circ^{\Phi\mathcal{A}}),$$

where

$$d[m_{a,b}] = [\mathrm{id}_{a,\phi_0^{\mathcal{A}}(a),0}], \quad d[\mathrm{id}_{a,b,i}] = [\mathrm{id}_{a,b,i}],$$

$$c[m_{a,b}] = [\mathrm{id}_{a,\phi_0^{\mathcal{A}}(a),1}], \quad c[\mathrm{id}_{a,b,i}] = [\mathrm{id}_{a,b,i}]$$

for every $a, b \in A$, $i \in 2$.

The operation $x \circ y$ is to be defined iff $d(x) = c(y)$. The interesting case is $x = m_{a,b}$, $y = m_{c,d}$. In this case $d(x) = c(y)$ iff $a = c$ and $\phi_0^{\mathcal{A}}(a) = a$. Let

$$\begin{aligned}
[\mathrm{id}_{a,b,i}] \circ [\mathrm{id}_{a,b,i}] &= [\mathrm{id}_{a,b,i}], \\
[m_{a,b}] \circ [\mathrm{id}_{a,\phi_0^{\mathcal{A}}(a),0}] &= [m_{a,b}], \\
[\mathrm{id}_{a,\phi_0^{\mathcal{A}}(a),1}] \circ [m_{a,b}] &= [m_{a,b}], \\
[m_{a,b}] \circ [m_{a,c}] &= [m_{a,\phi_1^{\mathcal{A}}(a)}], \qquad \text{if } \phi_0^{\mathcal{A}}(a) = a,
\end{aligned}$$

where $a, b \in A$, $i \in 2$.

It is straightforward to verify that the equations from E are satisfied and that $Ff : \Phi\mathcal{A} \to \Phi\mathcal{B}$ is a **Fix**(2)-morphism whenever $f : \mathcal{A} \to \mathcal{B}$ is a **Cat**-morphism. Hence Φ is a functor.

To prove that (Φ, F) is an s-embedding, let $\mathcal{A} = (A, (\phi_i^{\mathcal{A}})_{i\in 2}), \mathcal{B} = (B, (\phi_i^{\mathcal{B}})_{i\in 2}) \in$ **Fix**(2) and $f : FA \to FB$ be a **Cat**-homomorphism $\Phi\mathcal{A} \to \Phi\mathcal{B}$. For every $a \in A$ we have

$$[\mathrm{id}_{f(a),f(\phi_0^{\mathcal{A}}(a)),0}] = Ff(d[m_{a,a}]) = d(Ff[m_{a,a}]) = [\mathrm{id}_{f(a),\phi_0^{\mathcal{B}}(f(a)),0}],$$

hence $f(\phi_0^{\mathcal{A}}(a)) = \phi_0^{\mathcal{B}}(f(a))$.

For every $a \in A$ such that $\phi_0^{\mathcal{A}}(a) = a$ we have

$$[m_{f(a),f(\phi_1^{\mathcal{A}}(a))}] = Ff([m_{a,a}] \circ [m_{a,a}]) = Ff[m_{a,a}] \circ Ff[m_{a,a}] = [m_{f(a),\phi_1^{\mathcal{B}}(f(a))}],$$

hence $f(\phi_1^{\mathcal{A}}(a)) = \phi_1^{\mathcal{B}}(f(a))$. Therefore $f : \mathcal{A} \to \mathcal{B}$ is a **Cat**-morphism and the proof is concluded. ☐

5 Closure rules

The following formalization of the "properties which are inherited to slices" was suggested by J. Sichler in an unpublished note.

Definition 6 A triple $\mathtt{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$ is called a *closure rule*, if $\mathbf{a}_i$ $(i = 0, 1, 2)$ are small categories with the same set of objects, $\mathbf{a}_0$ is a subcategory of $\mathbf{a}_1$ and $\mathbf{a}_1$ is a subcategory of $\mathbf{a}_2$.

Definition 7 Let $\mathtt{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$ be a closure rule and $i_0 : \mathbf{a}_0 \to \mathbf{a}_1$ and $i_1 : \mathbf{a}_1 \to \mathbf{a}_2$ denote the inclusion functors. We say, that a faithful functor $U : \mathbf{K} \to \mathbf{H}$ *obeys* $\mathtt{a}$, if for every pair of functors $G_0 : \mathbf{a}_0 \to \mathbf{K}$, $G_2 : \mathbf{a}_2 \to \mathbf{H}$ such that $G_2 i_1 i_0 = U G_0$, there exists a functor $G_1 : \mathbf{a}_1 \to \mathbf{K}$ such that $G_1 i_0 = G_0$ and $U G_1 = G_2 i_1$. Notation: $U \vDash \mathtt{a}$.

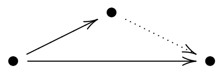
$$\begin{tikzcd}
\mathbf{a}_0 \arrow[r, hook, "i_0"] \arrow[d, "G_0"'] & \mathbf{a}_1 \arrow[r, hook, "i_1"] \arrow[dl, dashed, "G_1"] & \mathbf{a}_2 \arrow[d, "G_2"] \\
\mathbf{K} \arrow[rr, "U"'] & & \mathbf{H}
\end{tikzcd} \tag{4}$$

A closure rule $\mathtt{a}$ is said to be *trivial* provided that $U \vDash \mathtt{a}$ for every faithful functor U.

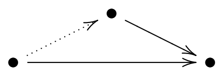
Let $\mathtt{a}, \mathtt{b}$ be closure rules. We say that $\mathtt{b}$ is a (semantic) consequence of $\mathtt{a}$, if $U \vDash \mathtt{a}$ implies $U \vDash \mathtt{b}$ for every faithful functor U. Notation: $\mathtt{a} \vDash \mathtt{b}$.

All closure rules used in this paper have the property that the category $\mathbf{a}_2$ is a quasiordered set, i.e. there is at most one arrow between any two objects of $\mathbf{a}_2$.

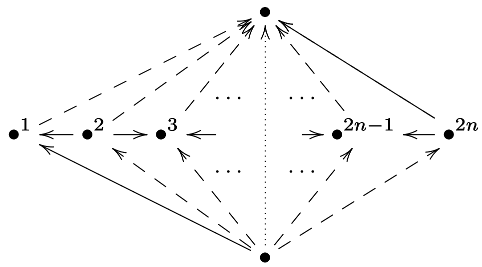
The following closure rules play an important role for the baskets in Figure 1 (see Introduction).



(p)



(p^{op})



(zz^1_n), $n \geq 1$

The nodes in the picture denote elements of the common set of objects of the closure rule. Arrows are $\mathbf{a}_2$-morphisms (identities are not drawn), solid arrows are $\mathbf{a}_0$-morphisms and dotted arrows are $\mathbf{a}_1$-morphisms.

Let $U : \mathbf{K} \to \mathbf{H}$ be a concrete category. The definition of $U \vDash \mathsf{a}$ says the following: Whenever we have objects of $\mathbf{K}$ and $\mathbf{H}$-morphisms between the respective underlying $\mathbf{H}$-objects, as in the picture, such that the diagram is commutative and solid arrows are $\mathbf{K}$-morphisms, then the dotted arrows are $\mathbf{K}$-morphisms as well.

Remark 4 1. It can be readily seen that a faithful functor $U : \mathbf{K} \to \mathbf{H}$ obeys each of closure rules $\mathsf{a}^i = (\mathbf{a}_0^i, \mathbf{a}_1^i, \mathbf{a}_2^i), i \in I$ iff U obeys its coproduct

$$\coprod_{i \in I} \mathsf{a}^i = (\coprod_{i \in I} \mathbf{a}_0^i, \coprod_{i \in I} \mathbf{a}_1^i, \coprod_{i \in I} \mathbf{a}_2^i).$$

By (zz^1) is meant the coproduct of the closure rules (zz^1_n).

2. A faithful functor $U : \mathbf{K} \to \mathbf{H}$ obeys a closure rule $\mathsf{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$ iff U^{op} obeys the dual closure rule $\mathsf{a}^{op} = (\mathbf{a}_0^{op}, \mathbf{a}_1^{op}, \mathbf{a}_2^{op})$.
3. It can be easily checked that the (forgetful functor of the) category of algebras with one nullary operation obeys (p), and the category of algebras with one unary operation obeys (zz^1) (this fact is a special case of Proposition 6). Obviously $(p) \vDash (zz^1)$, $(p^{op}) \vDash (zz^1)$ and $(zz^1_{n+1}) \vDash (zz^1_n)$.

If a faithful functor U obeys a closure rule a, then so does every slice of U:

Proposition 5 *Let $U : \mathbf{K} \to \mathbf{H}$, $U' : \mathbf{K}' \to \mathbf{H}'$ be concrete categories, a be a closure rule. If $U \leq_s U'$ and $U' \vDash \mathsf{a}$, then $U \vDash \mathsf{a}$.*

Proof Let be G_0, G_2 be functors such that diagram (4) is commutative, (Φ, F) be an s-embedding of U to U'. Let $A, B \in \mathrm{Obj}(\mathbf{a}_0)$ and $f \in \mathbf{a}_1(A, B)$ (dotted arrow). Since U' obeys a, $FG_2 f$ is a $\mathbf{K}$-morphism from $\Phi G_0 A$ to $\Phi G_0 B$, hence $G_2 f$ is a $\mathbf{K}$-morphism from $G_0 A$ to $G_0 B$, because $U \leq_s U'$. □

Remark 5 1. An easy consequence of Proposition 5 is that s-equivalent faithful functors obey the same closure rules. Therefore the formulation "the basket ... obeys ..." makes sense. From Remark 4 it follows that $\mathbb{P} \vDash (p)$, $\mathbb{P}^{op} \vDash (p)^{op}$, $\mathbb{A} \vDash (zz^1)$.

2. Proposition 5 enables us to show that certain s-inequality $U \leq_s U'$ doesn't hold: It suffices to find a closure rule which is obeyed by U' but it is not obeyed by U.
3. The notion of a closure rule could be generalized and Proposition 5 would remain true. For instance, consider a concrete category $U : \mathbf{K} \to \mathbf{H}$. The condition "the composition of two $\mathbf{H}$-morphism which are not $\mathbf{K}$-morphisms is not a $\mathbf{K}$-morphism" inherits also to slices of U. However we have no application of such generalizations.

Now we are going to define inductively closure rules (zz^α) (for every ordinal α) which are obeyed by essentially algebraic categories of height α.

Definition 8 Let $U : \mathbf{K} \to \mathbf{H}$ be a concrete category, A, B be $\mathbf{K}$-objects, $f \in \mathbf{H}(A, B)$.

– f is called (zz^0)-morphism.
– Let α be an ordinal; f is said to be a (zz^{α^+})-morphism, if there exists a commutative diagram

$$\begin{tikzcd}
 & & & B & & \\
C_1 \arrow[urrr, dashed, "h_1"] & C_2 \arrow[l, "l_1"'] \arrow[r, "l_2"] \arrow[urr, dashed, "h_2"] & C_3 \arrow[ur, dashed, "h_3"] & \cdots \arrow[l, dashed] \arrow[r, dashed] & C_{2n-1} \arrow[ul, dashed] & C_{2n} \arrow[l, "l_{2n-1}"'] \arrow[ull, "h_{2n}"'] \\
 & & & A \arrow[uu, dotted, "f"] \arrow[ulll, "g_1"] \arrow[ull, dashed, "g_2"] \arrow[ul, dashed, "g_3"] \arrow[ur, dashed, "g_{2n-1}"] \arrow[urr, dashed, "g_{2n}"'] & &
\end{tikzcd}$$

(zz^α)

where points are $\mathbf{K}$-objects, all arrows are $\mathbf{H}$-morphisms, solid arrows are $\mathbf{K}$-morphisms and dashed double arrows are (zz^α)-morphisms.
– Let α be a limit ordinal; f is said to be a (zz^α)-morphism, if it is a (zz^β)-morphism for every $\beta < \alpha$.

We say that U obeys (zz^α), if every (zz^α)-morphism is a $\mathbf{K}$-morphism.

Remark 6 1. For any α, every $\mathbf{K}$-morphism is a (zz^α)-morphism.
2. Note that (zz^α) can be written in the form of a closure rule. The rule (zz^1) coincides with the earlier defined version. If $\alpha \leq \beta$, then $(zz^\alpha) \vDash (zz^\beta)$.
3. It can be easily verified that the composition of a (zz^α)-morphism and a (zz^β)-morphism is a $(zz^{\min(\alpha,\beta)})$-morphism. In particular (zz^α)-morphisms are closed under composition.

Proposition 6 *Let α be an ordinal. Let $\mathbf{K}$ be an essential algebraic category of height α with any of the two natural forgetful functors. Then $\mathbf{K} \vDash (zz^\alpha)$. In particular $\mathbb{E}_\alpha \vDash (zz^\alpha)$ and dually $\mathbb{E}_\alpha^{op} \vDash (zz^\alpha)^{op}$.*

Proof Since both forgetful functors of $\mathbf{K}$ are slices of $\mathbf{Fix}(\alpha)$ (Theorem 1), it suffices to prove $\mathbf{Fix}(\alpha) \vDash (zz^\alpha)$. We proof by induction on $\beta \leq \alpha$ that every (zz^β)-morphism $f : \mathcal{A} = (A, (\phi_\gamma^{\mathcal{A}})_{\gamma<\alpha}) \to \mathcal{B} = (B, (\phi_\gamma^{\mathcal{B}})_{\gamma<\alpha})$ is a $\mathbf{Fix}(\beta)$-morphism $(A, (\phi_\gamma^{\mathcal{A}})_{\gamma<\beta}) \to (B, (\phi_\gamma^{\mathcal{B}})_{\gamma<\beta})$.

For $\beta = 0$ the statement is empty, for limit β it is clear. Now we assume that the statement holds for β and we will prove it for β^+. Since f is a (zz^{β^+})-morphism, we can find $\mathbf{Fix}(\beta^+)$-objects $\mathcal{C}_i$ and mappings g_i, h_i, l_i as in the diagram in Definition 8.

From the induction hypothesis we know that f preserves the operations ϕ_γ for all $\gamma < \beta$. Let $a \in A$ be in the definition domain of $\phi_\beta^{\mathcal{A}}$. We have

$$\begin{aligned}
f\phi_\beta^{\mathcal{A}}(a) &= h_1 g_1 \phi_\beta^{\mathcal{A}}(a) = \quad [g_1 \text{ is a } \mathbf{Fix}(\beta)\text{-morphism}] \\
&= h_1 \phi_\beta^{\mathcal{C}_1} g_1(a) = \\
&= h_1 \phi_\beta^{\mathcal{C}_1} l_1 g_2(a) = \ [g_2 \text{ is a } (zz^\beta)\text{-morphism and } l_1 \text{ a } \mathbf{Fix}(\beta)\text{-morphism}] \\
&= h_1 l_1 \phi_\beta^{\mathcal{C}_2} g_2(a) = \\
&= h_3 l_2 \phi_\beta^{\mathcal{C}_2} g_2(a) = \\
&= h_3 \phi_\beta^{\mathcal{C}_3} l_2 g_2(a) = h_3 \phi_\beta^{\mathcal{C}_3} l_3 g_4(a) = \\
&\quad \dots \\
&= h_{2n} \phi_\beta^{\mathcal{C}_{2n}} g_{2n}(a) = \phi_\beta^{\mathcal{B}} h_{2n} g_{2n}(a) = \phi_\beta^{\mathcal{B}} f(a).
\end{aligned}$$

$\square$

Proposition 7 *Let α be an ordinal. Then $\mathbb{E}_\alpha \neq \mathbb{E}_{\alpha^+}$.*

Proof According to Propositions 5, 6 it suffices to construct a mapping between algebras in $\mathbf{Fix}(\alpha^+)$ which is not a $\mathbf{Fix}(\alpha^+)$-homomorphisms, but it is a (zz^α)-morphism.

Let $\mathcal{A} = (1, (\phi_\gamma^{\mathcal{A}})_{\gamma \leq \alpha})$ be the unique $\mathbf{Fix}(\alpha^+)$-algebra on the set 1. For $0 \leq \beta \leq \alpha$, let $\mathcal{B}_\beta = (2, (\phi_\gamma^{\mathcal{B}_\beta})_{\gamma \leq \alpha})$, where

$$\phi_\gamma^{\mathcal{B}_\beta}(i) = \begin{cases} i & \text{if } \gamma < \beta,\ i \in 2 \\ 1 - i & \text{if } \gamma = \beta,\ i \in 2 \\ \text{undefined otherwise} & \end{cases}.$$

In what follows, c_0 denotes the constant mapping with the value 0 (domains and codomains vary). The mapping $c_0 : \mathcal{A} \to \mathcal{B}_{\alpha^+}$ is not a homomorphism, because it doesn't preserve the operation ϕ_α. We will show by induction on $\beta \leq \alpha$ that $c_0 : \mathcal{A} \to \mathcal{B}_\beta$ is a (zz^β)-morphism.

First step: Every mapping is a (zz^0)-morphism.

Isolated step: Suppose that $c_0 : \mathcal{A} \to \mathcal{B}_\beta$ is a (zz^β)-morphism. The following diagram shows that $c_0 : \mathcal{A} \to \mathcal{B}_{\beta^+}$ is a (zz^{β^+})-morphism

$$\begin{tikzcd}
 & \mathcal{B}_{\beta+} & \\
\mathcal{A} \arrow[ur, dashed, "c_0"] & & \mathcal{B}_\beta \arrow[ul, "c_0"'] \arrow[ll, "c_0"'] \\
 & \mathcal{A} \arrow[ul, "c_0"] \arrow[ur, Rightarrow, dashed, "c_0"'] &
\end{tikzcd}$$

Limit step: Suppose that $c_0 : \mathcal{A} \to \mathcal{B}_\delta$ is a (zz^δ)-morphism for every $\delta < \beta$, where β is a limit ordinal. Since $c_0 : \mathcal{B}_\delta \to \mathcal{B}_\beta$ is a $\mathbf{Fix}(\alpha^+)$-morphism, $c_0 : \mathcal{A} \to \mathcal{B}_\beta$ is a (zz^δ)-morphism following Remark 6.3. $\square$

Proposition 8 *Let α be an ordinal. Then $\mathbb{E}_2 \not\leq_s \mathbb{E}_\alpha^{op}$ (and dually $\mathbb{E}_2^{op} \not\leq_s \mathbb{E}_\alpha$).*

Proof According to Proposition 5 and Proposition 6, it suffices to construct a mapping between algebras in $\mathbf{Fix}(2)$ which is not a $\mathbf{Fix}(2)$-homomorphism, but it is a $(zz^\alpha)^{op}$-morphism.

Let $\mathcal{A} = (1, (\phi_i^{\mathcal{A}})_{i\in 2})$ be the $\mathbf{Fix}(2)$-algebra on the set 1. Let $\mathcal{B} = (2, (\phi_i^{\mathcal{B}})_{i\in 2})$ and $\mathcal{C} = (2, (\phi_i^{\mathcal{C}})_{i\in 2})$, where

$$\phi_0^{\mathcal{B}}(i) = 1 - i, \quad \phi_0^{\mathcal{C}}(i) = i, \quad \phi_1^{\mathcal{C}}(i) = 1 - i, \quad i \in 2.$$

Let c_0 be the constant mapping with the value 0 (domains and codomains vary again). The mapping $c_0 : \mathcal{A} \to \mathcal{C}$ is not a $\mathbf{Fix}(2)$-homomorphisms, since it doesn't preserve ϕ_1. By induction on β we prove that it is a $(zz^\beta)^{op}$-morphism $\mathcal{A} \to \mathcal{C}$.

First step: Every mapping is a $(zz^0)^{op}$-morphism.

Isolated step: Suppose that $c_0 : \mathcal{A} \to \mathcal{C}$ is a $(zz^\beta)^{op}$-morphism. The following diagram shows that it is a $(zz^{\beta+})^{op}$-morphism.

$$\begin{tikzcd}
& \mathcal{C} & \\
\mathcal{A} \arrow[ur, Rightarrow, dashed, "c_0"] & & \mathcal{B} \arrow[ul, "c_0"'] \arrow[ll, "c_0"'] \\
& \mathcal{A} \arrow[ul, "c_0"] \arrow[ur, dashed, "c_0"'] &
\end{tikzcd}$$

Limit step: Suppose that $c_0 : \mathcal{A} \to \mathcal{C}$ is a $(zz^\gamma)^{op}$-morphism for all $\gamma < \beta$. Then it is a $(zz^\beta)^{op}$-morphism. $\square$

The reasons for $(p) \vDash (zz^1_n), (zz^1_{n+1}) \vDash (zz^1_n)$ are syntactic – we can see it from the pictures of these closure rules. Theorem 2 bellow says that this is not by chance.

Definition 9 Let $\mathtt{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$, $\mathtt{b} = (\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)$ be closure rules. Let $\mathbf{b}$ be the smallest subcategory of $\mathbf{b}_2$, such that $\mathbf{b}_0 \subset \mathbf{b}$ and the functor $\subseteq : \mathbf{b} \to \mathbf{b}_2$ obeys $\mathtt{a}$. We say, that $\mathtt{b}$ is a syntactic consequence of $\mathtt{a}$, if $\mathbf{b}_1 \subset \mathbf{b}$. Notation: $\mathtt{a} \vdash \mathtt{b}$.

Remark 7 This smallest subcategory exists, it can be formed as the intersection of those satisfying the condition. This category can also be constructed by transfinite induction: We start with $\mathbf{b} = \mathbf{b}_0$. Then we repeat the following steps unless no new element can be added to $\mathbf{b}$ (at the limit step, we take the union, of course).

1. Take functors $H_0 : \mathbf{a}_0 \to \mathbf{b}$, $H_2 : \mathbf{a}_2 \to \mathbf{b}_2$ such that $jH_0 = H_2 i_1 i_0$, where j is the inclusion $j : \mathbf{b} \to \mathbf{b_2}$. Add all morphisms $H_2 f$ where f is a morphism of $\mathbf{a}_1$.
2. Make a closure of $\mathbf{b}$ with respect to composition.

It's clear that this leads to the category $\mathbf{b}$ from the definition.

Theorem 2 *Let $\mathtt{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$, $\mathtt{b} = (\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)$ be closure rules. Then $\mathtt{a} \vDash \mathtt{b}$ if and only if $\mathtt{a} \vdash \mathtt{b}$.*

Proof "$\Rightarrow$". Suppose $\mathtt{a} \nvdash \mathtt{b}$. Let $\mathbf{b}$ be the smallest subcategory from Definition 9. The concrete category $\subseteq : \mathbf{b} \to \mathbf{b}_2$ obeys $\mathtt{a}$ (according to the definition) and doesn't obey $\mathtt{b}$: Put $G_0 : \mathbf{b}_0 \to \mathbf{b}$ to be the inclusion and $G_2 : \mathbf{b}_2 \to \mathbf{b}_2$ to be the identity. Now G_1 from Definition 7 doesn't exist, since $\mathbf{b}_1 \not\subseteq \mathbf{b}$. Hence $\mathtt{a} \nvDash \mathtt{b}$.

"$\Leftarrow$". Assume $\mathtt{a} \vdash \mathtt{b}$ and let $U : \mathbf{K} \to \mathbf{H}$ be a concrete category which obeys $\mathtt{a}$. Striving for a contradiction, assume that U doesn't obey $\mathtt{b}$, i.e. there exist functors

$G_0 : \mathbf{b}_0 \to \mathbf{K}$, $G_2 : \mathbf{b}_2 \to \mathbf{H}$ such that $G_2 i_1 i_0 = UG_0$ and there is no functor $G_1 : \mathbf{b}_1 \to \mathbf{K}$ completing the commutative diagram (4). Let $\mathbf{b}'_0$ be the maximal subcategory of $\mathbf{b}_2$ such that there exists a functor $G'_0 : \mathbf{b}'_0 \to \mathbf{K}$ for which the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{b}_0 \arrow[r, hook] \arrow[d, "G_0"'] & \mathbf{b}'_0 \arrow[r, hook] \arrow[dl, "G'_0"'] & \mathbf{b}_2 \arrow[d, "G_2"] \\
\mathbf{K} \arrow[rr, "U"] & & \mathbf{H}
\end{tikzcd}$$

(The category $\mathbf{b}'_0$ thus consists precisely of those $\mathbf{b}_2$-morphisms $g : c \to d$ for which $G_2 g : G_0 c \to G_0 d$ is a $\mathbf{K}$-morphism.)

Since $\mathbf{b}_1 \not\subseteq \mathbf{b}'_0$ and $\mathsf{a} \vdash \mathsf{b}$, there exist functors $H_0 : \mathbf{a}_0 \to \mathbf{b}'_0$, $H_2 : \mathbf{a}_2 \to \mathbf{b}_2$ such that there is no $H_1 : \mathbf{a}_1 \to \mathbf{b}'_0$ for which the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{a}_0 \arrow[r, hook] \arrow[d, "H_0"'] & \mathbf{a}_1 \arrow[r, hook] \arrow[dl, "H_1"'] & \mathbf{a}_2 \arrow[d, "H_2"] \\
\mathbf{b}'_0 \arrow[rr, hook] & & \mathbf{b}_2
\end{tikzcd}$$

In other words, there are objects $c, d \in \mathrm{Obj}(\mathbf{a}_0)$ and $f \in \mathbf{a}_1(c, d)$ such that $H_2 f : H_0 c \to H_0 d$ isn't a $\mathbf{b}'_0$-morphism. But $G_2 H_2 f : G'_0 H_0 c \to G'_0 H_0 d$ is a $\mathbf{K}$-morphism, because $U \vDash \mathsf{a}$. This is a contradiction with the maximality of $\mathbf{b}'_0$. $\square$

An easy consequence of the proof is:

Corollary 3 *Let a, b be closure rules. Then $\mathsf{a} \vDash \mathsf{b}$, iff $U \vDash \mathsf{a}$ implies $U \vDash \mathsf{b}$ for all faithful functors U between small categories.*

Corollary 4 *A closure rule a is trivial iff $\mathbf{a}_0 = \mathbf{a}_1$.*

Proof If $\mathbf{a}_0 = \mathbf{a}_1$, then a is clearly trivial.

If $\mathbf{a}_0 \neq \mathbf{a}_1$, then the concrete category $\subseteq : \mathbf{a}_0 \to \mathbf{a}_2$ doesn't obey a. $\square$

6 Universality with respect to closure rules

First recall relevant known results:

Theorem 3 *Let $U : \mathbf{K} \to \mathbf{H}$ be a concrete category. Then*

1. $U \leq_s \mathbb{R}$ iff U is SSF (see [10]).
2. $U \leq_s \mathbb{P}$ iff U is SSF and $U \vDash (p)$ (see [10]).
2^{op}. $U \leq_s \mathbb{P}^{op}$ iff U is SSF and $U \vDash (p^{op})$.
3. Let both $\mathbf{K}, \mathbf{H}$ be small; or $\mathbf{H} = \mathbf{Set}$. $U \leq_s \mathbb{A}$ iff U is SSF and $U \vDash (zz^1)$. (see [9] for the small case, [8] for the set case)

The following theorem characterizes small slices of $\mathbb{E}_\alpha$.

Theorem 4 *Let $U : \mathbf{k} \to \mathbf{h}$ be a concrete category, where $\mathbf{k}$, $\mathbf{h}$ are small. Then $U \leq_s \mathbb{E}_\alpha$ iff $U \vDash (zz^\alpha)$.*

Proof If $U \leq_s \mathbb{E}_\alpha$ then $U \vDash (zz^\alpha)$ follows from Propositions 5, 6.

Suppose that $U \vDash (zz^\alpha)$. We will find an s-embedding (Φ, F) from U to $\mathbf{Fix}(P)$, where the poset P is the ordinal α plus a second minimal element $\overline{0}$:

$$P = \alpha \sqcup \{\overline{0}\}, \quad \overline{0} < \beta \text{ iff } 0 < \beta.$$

This is enough due to Theorem 1 (just a small part is needed).

First we define, for every $H \in \mathrm{Obj}(\mathbf{h})$ and $0 \leq \beta < \alpha$, a set $G^\beta H$ and an equivalence $\approx_\beta$ on $G^\beta H$:

$$\begin{aligned} G^\beta H = \{(A, g, B, h, \beta) \mid\ & A, B \in \mathrm{Obj}(\mathbf{k}),\ g \in \mathbf{h}(A, B) \text{ is a } (zz^\beta)\text{-morphism}, \\ & h \in \mathbf{h}(B, H)\}. \end{aligned}$$

The equivalence $\approx_\beta$ is given by

$$(A, g, B, h, \beta) \approx_\beta (A, g', B', h', \beta)$$

iff there exists a commutative diagram

$$\begin{tikzcd}
 & & & H & & \\
B \arrow[urrr, dashed, "h"] & B_2 \arrow[l] \arrow[r] \arrow[urr, dashed] & B_3 \arrow[ur, dashed] & {} \arrow[l] \arrow[r] & B_{2n-1} \arrow[ul, dashed] & B' \arrow[l] \arrow[ulll, dashed, "h'"'] , \\
 & & & A \arrow[ulll, Rightarrow, dashed, "g"'] \arrow[ull, Rightarrow, dashed] \arrow[ul, Rightarrow, dashed] \arrow[ur, Rightarrow, dashed] \arrow[urr, Rightarrow, dashed, "g'"'] & &
\end{tikzcd}$$

where B_i are $\mathbf{k}$ objects, arrows are $\mathbf{h}$-morphisms (between the respective objects), solid arrows are $\mathbf{k}$-morphisms and dashed double arrows are (zz^β)-morphisms.

The functor F is defined for $\mathbf{h}$-objects H, H' and $f \in \mathbf{h}(H, H')$ as follows:

$$\begin{aligned} FH &= \{[A, g, B, h, \beta]_{\approx_\beta} \mid \beta < \alpha,\ (A, g, B, h, \beta) \in G^\beta H, \}/\approx, \\ Ff[A, g, B, h, \beta]_\approx &= [A, g, B, fh, \beta]_\approx, \end{aligned}$$

where the equivalence $\approx$ is given by

$$[A, g, B, h, \beta]_{\approx_\beta} \approx [A, g', B', h', \beta']_{\approx_{\beta'}}$$

iff

$$(A, g, B, h, \beta) \approx_\beta (A, \mathrm{id}_{UA}, A, hg, \beta) \text{ and } (A, g', B', h', \beta') \approx_{\beta'} (A, \mathrm{id}_{UA}, A, hg, \beta').$$

In what follows we omit the subscript $\approx$.

We will use the following abbreviation:

$$(A, g) = [A, \mathrm{id}_{UA}, A, g, \beta],$$

where β is arbitrary (the right hand side does not depend on β). Note that $Ff(A, g) = (A, fg)$.

Observe that

- $Ff[A,g,B,h,\beta]$ doesn't depend on the choice of the representative and that F preserves the composition and identities. Thus F is a correctly defined functor $F : \mathbf{h} \to \mathbf{Set}$.
- Let A, K be $\mathbf{k}$-objects, $g \in \mathbf{h}(A,K)$ be a (zz^{β})-morphism. Then $(A,g) = [A,g,K,\mathrm{id}_{UK},\beta]$ iff g is a (zz^{β^+})-morphism.

Next we define the functor Φ. Let $K \in \mathrm{Obj}(\mathbf{k})$.

$$\Phi K = (FUK, (\phi_p^{\Phi K})_{p\in P}),$$

where the total operations $\phi_0^{\Phi K}$, $\phi_{\overline{0}}^{\Phi K}$ are given by

$$\phi_{\overline{0}}^{\Phi K}[A,f,B,g,\beta] = (A,gf),$$
$$\phi_0^{\Phi K}[A,f,B,g,\beta] = [A,gf,K,\mathrm{id}_{UK},0].$$

Let $0 < \beta < \alpha$. The operation $\phi_\beta^{\Phi K}$ is defined by

$$\mathrm{Def}(\phi_\beta^{\Phi K}) = \{(A,g) \,|\, g : A \to K \text{ is a } (zz^{\beta})\text{-morphism } \},$$
$$\phi_\beta^{\Phi K}(A,g) = [A,g,K,\mathrm{id}_{UK},\beta].$$

To verify that ΦK is a $\mathbf{Fix}(P)$-object, we have to check the following:

Claim Let $0 < \beta < \alpha$. Then $\mathrm{Fix}(\{\phi_p^{\Phi K} \,|\, p < \beta\}) = \mathrm{Def}(\phi_\beta^{\Phi K})$.

Proof We proceed by induction on β.

First step: An element $x \in FUK$ is a fix-point of $\phi_{\overline{0}}^{\Phi K}$ iff $x = (A,g)$ for some $g \in \mathbf{H}(A,K)$. An element $(A,g) \in FUK$ is a fix-point of $\phi_0^{\Phi K}$ iff

$$(A,g) = [A,\mathrm{id}_{UA},A,g,0] = \phi_0^{\Phi K}[A,\mathrm{id}_{UA},A,g,0] = [A,g,K,\mathrm{id}_{UK},0].$$

This happens precisely when $g : A \to K$ is a (zz^1)-morphism.

Isolated step: Assume that $\mathrm{Fix}(\{\phi_p^{\Phi K} \,|\, p < \beta\}) = \mathrm{Def}(\phi_\beta^{\Phi K})$. The element (A,g), where $g \in \mathbf{H}(A,K)$ is a (zz^{β})-morphism, is a fix-point of $\phi_\beta^{\Phi K}$ iff

$$(A,g) = \phi_\beta^{\Phi K}(A,g) = [A,g,K,\mathrm{id}_{UK},\beta].$$

This happens precisely when g is a (zz^{β^+})-morphism $A \to K$ (see the observation above).

The limit step is obvious. $\square$

It is clear that Φ preserves the composition and identities. Therefore, to prove that Φ is a functor, we have to verify the following:

Claim Let $f : K \to L$ be a $\mathbf{k}$-morphism. Then $Ff : \Phi K \to \Phi L$ is a $\mathbf{Fix}(P)$-morphism.

Proof Ff preserves $\phi_{\overline{0}}$:

$$Ff(\phi_{\overline{0}}^{\Phi K}[A,g,B,h,\beta]) = Ff(A,hg) = (A,fhg),$$

$$\phi_{\overline{0}}^{\Phi L}(Ff[A,g,B,h,\beta]) = \phi_{\overline{0}}^{\Phi L}[A,g,B,fh,\beta] = (A,fhg).$$

Ff preserves ϕ_0:

$$Ff(\phi_0^{\Phi K}[A,g,B,h,\beta]) = Ff[A,hg,K,\mathrm{id}_{UK},0] = [A,hg,K,f,0],$$

$$\phi_0^{\Phi L}(Ff[A,g,B,h,\beta]) = \phi_0^{\Phi L}[A,g,B,fh,\beta] = [A,fhg,L,\mathrm{id}_{UL},0].$$

The right hand sides are equal, since $(A,hg,K,f,0) \approx_0 (A,fgh,L,\mathrm{id}_{UL},0)$:

$$\begin{tikzcd}
 & UL & \\
K \arrow[ur, dashed, "f"] \arrow[rr, "f"] & & L \arrow[ul, dashed, "\mathrm{id}_{UL}"'] \\
 & A \arrow[ul, dashed, "hg"] \arrow[ur, dashed, "hgf"'] &
\end{tikzcd}$$

Ff preserves ϕ_β, $0 < \beta < \alpha$: Let $g : A \to K$ be a (zz^β)-morphism. Then

$$Ff(\phi_\beta^{\Phi K}(A,g)) = Ff[A,g,K,\mathrm{id}_{UK},\beta] = [A,g,K,f,\beta],$$

$$\phi_\beta^{\Phi K}(Ff(A,g)) = \phi_\beta^{\Phi L}(A,fg) = [A,fg,L,\mathrm{id}_{UL},\beta].$$

The right hand sides are equal, since $(A,g,K,f,\beta) \approx_\beta (A,fg,L,\mathrm{id}_{UL},\beta)$:

$$\begin{tikzcd}
 & UL & \\
K \arrow[ur, dashed, "f"] \arrow[rr, "f"] & & L \arrow[ul, dashed, "\mathrm{id}_{UL}"'] \\
 & A \arrow[ul, Rightarrow, dashed, "g"] \arrow[ur, Rightarrow, dashed, "fg"'] &
\end{tikzcd}$$

(the dashed double arrows are (zz^β)-morphisms). $\square$

Finally, to show that (Φ, F) is an s-embedding, we prove

Claim Let $f : K \to L$ be a $\mathbf{h}$-morphism such that $Ff : \Phi K \to \Phi L$ is a $\mathbf{Fix}(\alpha)$-morphism. Then f is a $\mathbf{k}$-morphism.

Proof Let $\beta < \alpha$. The identity $\mathrm{id}_{UK} : K \to K$ is a $\mathbf{k}$-morphism, hence it is a (zz^β)-morphism. Thus $\phi_\beta^{\Phi K}(K, \mathrm{id}_{UK})$ is defined. We have

$$Ff(\phi_\beta^{\Phi K}(K,\mathrm{id}_{UK})) = Ff(K,\mathrm{id}_{UK}) = (K,f),$$

$$\phi_\beta^{\Phi L}(Ff(K,\mathrm{id}_{UK})) = \phi_\beta^{\Phi L}(K,f) = [K,f,L,\mathrm{id}_{UL},\beta].$$

Thus $(K,f) = [K,f,L,\mathrm{id}_{UL},\beta]$, hence $f : K \to L$ is a (zz^{β^+})-morphism.

Since $f : K \to L$ is a (zz^{β^+})-morphism for all $\beta < \alpha$, it is a (zz^α)-morphism. Because $U \models (zz^\alpha)$, we have $f \in \mathbf{K}(K,L)$. $\square$

The proof of Theorem 4 is concluded. $\square$

A consequence of Theorems 2, 3, 4 is that, loosely speaking, our baskets obey no other closure rule than we already know:

Corollary 5 *Let $\mathfrak{a}$ be a closure rule. Then*

1. $\mathbb{R} \models \mathsf{a}$ *iff* a *is trivial.*
2. $\mathbb{P} \models \mathsf{a}$ *iff* $(p) \vdash \mathsf{a}$*.*
3. $\mathbb{E}_\alpha \models \mathsf{a}$ *iff* $(zz^\alpha) \vdash \mathsf{a}$*.*

Proof We are going to prove 3. The remaining cases can be proved similarly.

If $(zz^\alpha) \vdash \mathsf{a}$, then $(zz^\alpha) \models \mathsf{a}$ (Theorem 2), whence $\mathbb{E}_\alpha \models \mathsf{a}$.

If $\mathbb{E}_\alpha \models \mathsf{a}$ and $U \models (zz^\alpha)$, where U is a faithful functor between small categories, then $U \leq_s \mathbb{E}_\alpha$ (Theorem 4), hence $U \models \mathsf{a}$. Thus $(zz^\alpha) \vdash \mathsf{a}$ due to Corollary 3, Theorem 2. $\square$

Using a modification of the last proof, we are able to give a slight generalization of Theorem 3.3. To formulate this result, we need the following definition.

Definition 10 We say that a category $\mathbf{H}$ satisfies (*), if for every $H \in \mathrm{Obj}(\mathbf{H})$ the following equivalence on the class of all morphism with codomain H has set-many equivalence classes only

$$f : A \to H \sim_* g : B \to H \quad \text{iff} \quad (\exists k : A \to B)\,(\exists l : B \to A)\ gk = f \text{ and } fl = g.$$

Theorem 5 *Let* $U : \mathbf{K} \to \mathbf{H}$ *be a concrete category, where both* $\mathbf{H}$ *and* $\mathbf{H}^{op}$ *satisfy (*). Then* $U \leq_s \mathbb{A}$ *iff* U *is SSF and obeys* (zz^1)*.*

Proof If $U \leq_s \mathbb{A}$, then U is SSF and $U \models (zz^1)$ (see Propositions 2, 6, 5).

Suppose that U is SSF and obeys (zz^1). We will find functors $F, G : \mathbf{H} \to \mathbf{Set}$ and a concrete full embedding $\Phi : \mathbf{K} \to \mathbf{A}[F,G]$. This suffices, since $\mathbf{A}[F,G] \leq_s \mathbb{A}$ (Let $C = F \sqcup G$, $\Psi(H,\alpha) = (FH \sqcup GH, \overline{\alpha})$, where $\overline{\alpha}$ coincides with α on FH and is identical on GH. (Ψ, C) is an s-embedding.)

For an $\mathbf{H}$-object, let

$$FH = \{(A,g) \mid A \in \mathrm{Obj}(\mathbf{K}),\ g \in \mathbf{H}(A,H)\}/\approx,$$

where

$$(A,g) \approx (A',g') \quad \text{iff} \quad (A,g) \sim_{SSF} (A',g') \text{ and } g \sim_* g'.$$

For a $\mathbf{H}$-morphism $f : H \to H'$ let

$$Ff[A,g]_\approx = [A, fg]_\approx.$$

Since U is SSF and $\mathbf{H}$ satisfy (*), there is set-many equivalence classes of $\approx$ only (for each H).

For $H \in \mathrm{Obj}(\mathbf{H})$ we now define a class $G'H$ by

$$G'H = \{(A,g,B,h) \mid A, B \in \mathrm{Obj}(\mathbf{K}),\ g \in \mathbf{H}(A,B),\ h \in \mathbf{H}(B,H)\}$$

and an equivalence $\approx_{zz}$ on $G'H$: $(A,g,B,h) \approx_{zz} (A',g',B',h')$, iff $(A,hg) \sim_{SSF} (A', h'g')$ and there exists a commutative diagram

$$\begin{tikzcd}
 & & & H & & \\
B \arrow[urrr, dashed, "h"] & B_2 \arrow[l] \arrow[r] \arrow[urr, dashed] & B_3 \arrow[ur, dashed] & \cdots \arrow[l] \arrow[r] & B_{2n-1} \arrow[ul, dashed] & B' \arrow[l] \arrow[ull, dashed, "h'"'] \\
A \arrow[u, dashed, "g"] \arrow[r, dashed, shift left] & A_2 \arrow[u, dashed] \arrow[l, dashed, shift left] \arrow[r, dashed, shift left] & A_3 \arrow[u, dashed] \arrow[l, dashed, shift left] & \cdots \arrow[l, dashed] \arrow[r, dashed] & A_{2n-1} \arrow[u, dashed] \arrow[r, dashed, shift left] & A' \arrow[u, dashed, "g'"'] \arrow[l, dashed, shift left]
\end{tikzcd}$$

where, again, A_i, B_i are **K**-objects, arrows are **H**-morphism and solid arrows are **K**-morphisms.

The functor $G : \mathbf{H} \to \mathbf{Set}$ is defined for $H, H' \in \mathrm{Obj}(\mathbf{H})$ and $f \in \mathbf{H}(H, H')$ by

$$GH = \{(A, g, B, h) \mid A, B \in \mathrm{Obj}(\mathbf{K}), g \in \mathbf{H}(A, B), h \in \mathbf{H}(B, H)\}/\equiv,$$
$$Gf[A, g, B, h]_{\equiv} = [A, g, B, fh]_{\equiv}.$$

The equivalence $\equiv$ is given by

$$(A, g, B, h) \equiv (A', g', B', h') \quad \text{iff} \quad \mathrm{OUT}(A, g, B, h) = \mathrm{OUT}(A', g', B', h'),$$

where

$$\mathrm{OUT}(A, g, B, h) = \{l \in \mathbf{H}(H, H') \mid H' \in \mathrm{Obj}(H),\ (\exists A' \in \mathrm{Obj}(\mathbf{K}))$$
$$(\exists m \in \mathbf{H}(A', H'))\ (A, g, B, lh) \approx_{zz} (A', \mathrm{id}_{UA'}, A', m)\}.$$

Observe that

- if $l \sim_* l'$ in $\mathbf{H}^{op}$, then $l \in \mathrm{OUT}(A, g, B, h)$ iff $l' \in \mathrm{OUT}(A, g, B, h)$. Thus $\equiv$ has set-many equivalence classes only.
- Let $C \in \mathrm{Obj}(\mathbf{K})$ be such that $UC = H$. If $\mathrm{id}_H \in \mathrm{OUT}(A, g, B, h)$ and $h \in \mathbf{K}(B, C)$, then $hg \in \mathbf{K}(A, C)$.

For each $K \in \mathrm{Obj}(K)$, let $\Phi K = (UK, \overline{K} : FUK \to GUK)$, where

$$\overline{K}[A, g]_{\approx} = [A, g, K, \mathrm{id}_{UK}]_{\equiv}.$$

The definition doesn't depend on the choice of the representative of $[A, g]$. Φ is a functor:

Claim Let $f : K \to L$ be a **K**-morphism. Then $f : \Phi K \to \Phi L$ is an $\mathbf{A}[F, G]$-morphism.

Proof Let $[A, g]_{\approx} \in FUK$. Then

$$Gf(\overline{K}[A, g]_{\approx}) = Ff[A, g, K, \mathrm{id}_{UK}]_{\equiv} = [A, g, K, f]_{\equiv}$$

and

$$\overline{L}(Ff[A, g]_{\approx}) = \overline{L}[A, fg]_{\approx} = [A, fg, L, \mathrm{id}_{UL}]_{\equiv}.$$

Since $(A, g, K, f) \approx_{zz} (A, gf, L, \mathrm{id}_{UL})$ it follows that $(A, g, K, f) \equiv (A, gf, L, \mathrm{id}_{UL})$. □

Φ is full:

Claim Let $f : K \to L$ be a **H**-morphism such that $f : \Phi K \to \Phi L$ is an $\mathbf{A}[F, G]$-morphism. Then f is a **K**-morphism from K to L.

Proof

$$Gf(\overline{K}[K, \mathrm{id}_{UK}]_{\approx}) = Gf[K, \mathrm{id}_{UK}, K, \mathrm{id}_{UK}]_{\equiv} = [K, \mathrm{id}_{UK}, K, f]_{\equiv}$$

and

$$\overline{L}(Ff[K, \mathrm{id}_{UK}]_{\approx}) = \overline{L}[K, f]_{\approx} = [K, f, L, \mathrm{id}_{UL}]_{\equiv}.$$

Since $\mathrm{id}_{UL} \in \mathrm{OUT}(K, \mathrm{id}_{UK}, K, f)$, $\mathrm{id}_{UL} \in \mathrm{OUT}(K, f, L, \mathrm{id}_{UL})$. Thus $f : K \to L$ is a **K**-morphism due to the observation above. □

The last claim finishes the proof of Theorem 5. □

The conditions on **H** are still very strong. However, they are satisfied by any small category, the category of sets, the category of pointed sets, the category of vector spaces.

Problem 2 Is it possible to generalize Theorem 4 to arbitrary concrete categories $U : \mathbf{K} \to \mathbf{H}$? Or, at least, answer the following particular questions:

– Is it possible to generalize Theorem 5 to arbitrary concrete categories $U : \mathbf{K} \to \mathbf{H}$? An attempt was made in author's master thesis: There exists a concrete category V such that $U \leq_s V$ iff U is SSF and $U \vDash (zz_1^1)$. This however doesn't seem to be the right direction.
– Is it possible to generalize Theorem 4 to concrete categories over **Set**? If the answer is positive, Theorem 1 would easily follow, since the usage of Theorem 1 in Proposition 6 is inessential.

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