Modularity and coloring of terms by variables

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Join of two varieties

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σ, ρ: their signatures

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It is the join in the lattice of interpretability types of varieties W. D. Neumann, 1974

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THEOREM A. Day 69 TFAE for a variety $\mathbb{V}$

- $\mathbb{V}$ is CM
- there are $t_1, t_2, \dots \in FX$ such that

$$x_0 \overset{\bar{\alpha},\bar{\beta}}{\sim} t_1 \overset{\bar{\gamma}}{\sim} t_2 \overset{\bar{\alpha},\bar{\beta}}{\sim} t_3 \overset{\bar{\gamma}}{\sim} \dots x_3$$

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- $(\forall \alpha \in A)\ (\forall t, s \in FX)\ F\alpha(t) = F\alpha(s) \ \Rightarrow\ \alpha(c(t)) = \alpha(c(s))$
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REMARK Similar to L. Sequeira's compatibilty with projections

A characterization of modularity

PROPOSITION TFAE

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REMARK Similar characterization for *n*-CP. Impossible for CD.

A-coloring of a set endofunctor

DEFINITION $F : \mathbf{Set} \to \mathbf{Set}$: functor s.t. $Id \subseteq F$, X: set, $A \subseteq X^X$.

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- A-coloring of a variety = A-coloring of its free functor

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REMARK

- A-coloring of a variety $=$ A-coloring of its free functor
- It's a weakening of natural transformation $F \to Id$

A-Coloring and composition

PROPOSITION $F, G : \mathbf{Set} \to \mathbf{Set}$: functors, X: set, $A \subseteq X^X$. If F, G are A-colorable, then FG is A-colorable.

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COROLLARY

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Compare

$\mathbb{V}$: idempotent variety
F: free functor of $\mathbb{V}$

Hobby, McKenzie 88: $\mathbb{V}$: locally finite

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Dualities!

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Thank you for your attention!