The complexity of the Quantified Constraint Satisfaction Problem on a 3-element set

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Quantified Constraint Satisfaction Problem

Let A be a finite set,
Γ be a set of of all predicates (or relations) on A, called constraint language

QCSP(Γ):

Given a sentence $\exists y_1 \forall x_1 \ldots \exists y_t \forall x_t (R_1(\ldots) \wedge \cdots \wedge R_s(\ldots))$, where $R_1, \ldots, R_s \in \Gamma$.
Decide whether it holds.

Examples

$A = \{0, 1, 2\}, \Gamma = \{x \neq y\}$. QCSP instances:

$\forall x \exists y_1 \exists y_2 (x \neq y_1 \wedge x \neq y_2 \wedge y_1 \neq y_2)$, true

$\forall x_1 \forall x_2 \forall x_3 \exists y (x_1 \neq y \wedge x_2 \neq y \wedge x_3 \neq y)$, false

$\forall x_1 \exists y_1 \forall x_2 \exists y_2 (x_1 \neq y_1 \wedge y_1 \neq y_2 \wedge y_2 \neq x_2)$, true

Constraint Satisfaction Problem

Let A be a finite set,
Γ be a set of of all predicates (or relations) on A.

CSP(Γ):

Given a formula $(R_1(\dots) \wedge \dots \wedge R_s(\dots))$,
where $R_1, \dots, R_s \in \Gamma$.
Decide whether the formula is satisfiable.

Constraint Satisfaction Problem

Let A be a finite set,
Γ be a set of of all predicates (or relations) on A.

CSP(Γ):

Given a sentence $\exists y_1 \dots \exists y_t (R_1(\dots) \wedge \dots \wedge R_s(\dots))$,
where $R_1, \dots, R_s \in \Gamma$.
Decide whether it holds.

Theorem [Bulatov, Zhuk, 2017]

- CSP(Γ) is solvable in polynomial time (tractable) if there exists a weak near-unanimity operation preserving Γ,
- CSP(Γ) is NP-complete otherwise.

Weak near-unanimity operation (WNU) is an operation satisfying

$$w(y, x, x, \dots, x) = w(x, y, x, \dots, x) = \dots = w(x, x, \dots, x, y)$$

Few facts about QCSP

- If Γ contains all predicates then QCSP(Γ) is PSpace-complete.
- If Γ consists of linear equations in a finite field then QCSP(Γ) can be solved in polynomial time (tractable).
- Put $A' = A \cup \{*\}$, Γ' is Γ extended to A'. Then QCSP(Γ') is equivalent to CSP(Γ).
- The complexity of QCSP(Γ) can be P, NP-complete, PSpace-complete. What else?

Main Question

What is the complexity of QCSP(Γ) for different Γ?

Easier problem

QCSP$^2(\Gamma)$:

Given a sentence $\forall x_1 \dots \forall x_t \exists y_1 \dots \exists y_q (R_1(\dots) \wedge \dots \wedge R_s(\dots))$, where $R_1, \dots, R_s \in \Gamma$.
Decide whether it holds.

- We need to check that for all evaluations of $x_1, \dots, x_t$ there exists a solution of the CSP $(R_1(\dots) \wedge \dots \wedge R_s(\dots))$.
- How many tuples it is sufficient to check?

PGP vs EGP

For an algebra $(A; F)$ (a set of operations F on a set A)
$d_F(n)$ is the minimal size of a generating set of A^n.

Examples

1. $A = \{0, 1\}$, $F = \{x \vee y\}$. $d_F(n) = n + 1$. It is sufficient to have $(0, \dots, 0)$ and $(0, \dots, 0, 1, 0, \dots, 0)$ for any position of 1 to generate $\{0, 1\}^n$.
2. $A = \{0, 1\}$, $F = \{\neg x\}$. $d_F(n) = 2^{n-1}$. It is sufficient to have all tuples starting with 0 to generate $\{0, 1\}^n$.

- If $d_F(n)$ is restricted by a polynomial in n, then the algebra has the Polynomially Generated Power (PGP) property
- If $d_F(n)$ is exponential in n, then the algebra has the Exponentially Generated Power (EGP) property

Theorem[Zhuk, 2015]

Every finite algebra either has PGP, or has EGP.

Easier Problem

$\mathrm{QCSP}^2(\Gamma)$:

Given a sentence $\forall x_1 \dots \forall x_t \exists y_1 \dots \exists y_q (R_1(\dots) \wedge \dots \wedge R_s(\dots))$, where $R_1, \dots, R_s \in \Gamma$.
Decide whether it holds.

Example

If Γ is preserved by $x \vee y$ then it is sufficient to check that $(R_1(\dots) \wedge \dots \wedge R_s(\dots))$ is satisfiable for $(x_1, \dots, x_t) = (0, \dots, 0)$ and $(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_t) = (0, \dots, 0, 1, 0, \dots, 0)$ for $\forall i$.

Theorem

If $\mathrm{Pol}(\Gamma)$ has PGP, then $\mathrm{QCSP}^2(\Gamma)$ can be polynomially reduced to $\mathrm{CSP}(\Gamma \cup \{x = a \mid a \in A\})$.

- $\mathrm{Pol}(\Gamma)$ is the set of all operations preserving Γ.

Theorem[B. Martin, C. Carvalho, F.Madelaine, D. Zhuk, 2017]

If $\mathrm{Pol}(\Gamma)$ has PGP, then $\mathrm{QCSP}(\Gamma)$ can be polynomially reduced to $\mathrm{CSP}(\Gamma \cup \{x = a \mid a \in A\})$.

Chen's Conjecture

Chen's Conjecture

If Pol(Γ) has EGP, then QCSP(Γ) is PSpace-complete.

QCSP Trichotomy Conjecture

QCSP(Γ)

- is tractable, if Pol(Γ) has PGP and WNU
- is NP-complete, if Pol(Γ) has PGP and has no WNU
- is PSpace-complete, if Pol(Γ) has EGP

Theorem[B.Martin, 2018]

The conjecture holds for Γ containing all unary predicates (the conservative case).

Demise of Chen's conjecture

- B. Martin and M. Olsak found Γ on 3-element domain such that QCSP(Γ) is coNP-complete.
- D.Zhuk found Γ on 4-element domain such that QCSP(Γ) is DP-complete, where $DP = NP \wedge coNP$.
- D.Zhuk found Γ on 10-element domain such that QCSP(Γ) is not tractable, not NP-complete, not coNP-complete, not DP-complete, not PSpace-complete.
- D.Zhuk found Γ having EGP such that QCSP(Γ) is tractable.

QCSP on 3-element domain

Theorem

Suppose Γ *is a constraint language on* $\{0, 1, 2\}$ *containing* $\{x = a \mid a \in \{0, 1, 2\}\}$. *Then* $\mathrm{QCSP}(\Gamma)$ *is*

- *tractable, or*
- *NP-complete, or*
- *coNP-complete, or*
- *PSpace-complete.*

QCSP on 3-element domain

Theorem

Suppose Γ *is a constraint language on* $\{0, 1, 2\}$ *containing* $\{x = a \mid a \in \{0, 1, 2\}\}$*. Then* $\mathrm{QCSP}(\Gamma)$ *is*

1. *tractable, if* $\mathrm{Pol}(\Gamma)$ *has PGP and has a WNU*
2. *NP-complete, if* $\mathrm{Pol}(\Gamma)$ *has PGP and has no WNU*
3. *PSpace-complete, if* $\mathrm{Pol}(\Gamma)$ *has EGP and has no WNU*
4. *PSpace-complete, if* $\mathrm{Pol}(\Gamma)$ *has EGP and* $\mathrm{Pol}(\Gamma)$ *does not contain* f *such that* $f(x, a) = x$ *and* $f(x, c) = c$*, where* $a, c \in \{0, 1, 2\}$*, then* $\mathrm{QCSP}(\Gamma)$
5. *tractable, if* $\mathrm{Pol}(\Gamma)$ *contains* $s_{a,c}$ *and* $g_{a,c}$ *for some* $a, c \in \{0, 1, 2\}$
6. *tractable, if* $\mathrm{Pol}(\Gamma)$ *contains* $f_{a,c}$ *for some* $a, c \in \{0, 1, 2\}$
7. *coNP-complete otherwise.*

New Tractable Cases

Counter-example to Chen's Conjecture

$$\Gamma = \left\{ \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & \cdot \\ 0 & 1 & 0 & 1 & \cdot & 2 \\ 0 & 0 & 1 & 1 & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & \cdot & \cdot \end{pmatrix} \right\}.$$

- Pol(Γ) has EGP.
- QCSP(Γ) can be solved in polynomial time.

Idea of the algorithm

- Reduce QCSP(Γ) to QCSP^2(Γ), i.e. $\forall x_1 \dots \forall x_t \exists y_1 \dots \exists y_q (R_1(\dots) \wedge \dots \wedge R_s(\dots))$.
- By solving CSP instances calculate a set of evaluations of $(x_1, \dots, x_t)$ we need to check.
- Check that $(R_1(\dots) \wedge \dots \wedge R_s(\dots))$ has a solution for each evaluation of $(x_1, \dots, x_t)$.

Open Question

What can be the complexity of QCSP(Γ)?

- for 3-element domain (nonidempotent case)
- for 4-element domain
- for bigger domains.

Thank you for your attention