Finitary set endofunctors are alg-universal

Libor Barto

Abstract. A category is said to be alg-universal, if every category of universal algebras can be fully embedded into it. We prove here that the category of finitary endofunctors of the category **Set** is alg-universal. We also present an example of a proper class of accessible set functors with no natural transformations between them (except the obvious identities).

1. Introduction

Every group is isomorphic to the monoid of all endotransformations of some endofunctor of **Set** (where **Set** denotes the category of all sets and mappings) – this was proved by P. Zima and the author, see [3]. Here we are going to prove a much stronger result: The category of finitary endofunctors of **Set** is alg-universal, i.e. every category of universal algebras can be fully embedded into it.

Let us recall related notions and results concerning representations in categories. The classical result of Birkhoff [4] about representations of groups as automorphism groups of complete distributive lattices was generalized to the investigation of full embeddings (i.e. functors which are bijective on hom-sets) of categories starting from [11] and [10].

We say that a category **K** is

group-universal,	if for every group G, there exists an object $A \in \mathrm{Obj}(\mathbf{K})$ s. t. $\mathrm{Aut}(A)$, the automorphism group of A, is isomorphic to G;
group-universal in a stronger sense,	if for every group G, there exists $A \in \mathrm{Obj}(\mathbf{K})$ s. t. $\mathrm{End}(A)$, the endomorphism monoid of A, is a group isomorphic to G;
monoid-universal,	if for every monoid M, there exists $A \in \mathrm{Obj}(\mathbf{K})$ s. t. $\mathrm{End}(A)$ is isomorphic to G;
alg-universal,	if every category of universal algebras can be fully embedded into $\mathbf{K}$;

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universal,	if every concretizable category (i.e. a category which admits a faithful functor into **Set**) can be fully embdedded into **K**;
hyper-universal,	if every category can be fully embdedded into **K**.

Every small category (in particular, a one object category – a monoid) can be fully embedded into some category of universal algebras (see [19]), hence every alg-universal category is monoid-universal. Alg-universality seems to be much stronger property than monoid-universality. However, no "natural" example (e.g. a variety or a quasivariety of algebras) of monoid-universal category which is not alg-universal is known. Kučera, Pultr and Hedrlín showed that the statement "every alg-universal category is universal" is equivalent to the following set-theoretical assumption: The class of all measurable cardinals is a set (see [19]). Every universal category has a factor (morphisms are glued together in an admissible way), which is hyper-universal (see [17, 21, 24]). No "natural" example of hyper-universal category is known.

A very long list of group-universal categories is presented in the survey paper [6] and all group-universal varieties of unary algebras were characterized in [20]. The category of (abstract) clones and clone homomorphisms [2], and the category of set functors ([3]) are group-universal in a stronger sense. However, the alg-universality seems to be the most important notion from the list above. In [10], the category **Rel**(2) of graphs and graph homomorphisms, and the category **Alg**$(1,1)$ of algebras with two unary operations and algebra homomorphisms were shown to be alg-universal. Then a lot of varieties of universal algebras were proved to be alg-universal, e.g. the variety of $(0,1)$-lattices [8], semigroups [9], integral domains of characteristic zero [5], and many others. These older results are summarized in the monograph [19] and in the survey article [25], where also many later results are mentioned, e.g. the full characterization of alg-universal varieties of $(0,1)$-lattices [7] and of semigroups [16].

There are also interesting universal categories, e.g. the category of hypergraphs (Hedrlín, Kučera, see [19]), the category of topological spaces and open continuous maps [19], the category of topological semigroups and continuous homomorphisms [24]. The regular varieties of topological unary algebras, which are universal, are characterized in [14].

The basic structural properties of set functors, i.e. endofunctors of the category **Set**, were obtained in the articles [22, 23, 13, 15]. The category of all set functors and all natural transformations is not legitimate, because there are "too many" set functors and "too many" natural transformations. But it has natural legitimate subcategories – the category of κ-accessible set functors for some cardinal κ and the category of accessible set functors. See section 2 for the definitions and preliminaries concerning set functors.

The category of finitary (ω-accessible) set functors and natural transformations is related to the category **Clone** of (abstract) clones and clone homomorphisms, or, in a different view, to the category of (finitary) varieties and interpretations.

Indeed, an interpretation between varieties can be viewed as a natural transformation between their free functors, which, in some sense, preserves equations. It turned out that our main theorem is the right direction to prove alg-universality of the category **Clone**. This result will appear in a forthcoming article.

Section 3 contains the proof of the main theorem of this paper: The category of finitary set functors is alg-universal. Since the category of κ-accessible set functors is algebraic for every κ (*algebraic* means here, that it can be fully embedded into some category of universal algebras), universality of this category is equivalent to the above mentioned set-theoretical assumption.

Recall that a class $\mathrm{Obj}(S)$ of objects in some category is said to be *rigid*, if $\mathrm{End}(A) = \{\mathrm{id}_A\}$ and $\mathrm{Hom}(A, B) = \emptyset$ for every $A \neq B \in \mathrm{Obj}(S)$. In any alg-universal category, there exists an arbitrarily large rigid set of objects, because we can embed arbitrarily large discrete (small) category. It turned out, that the statement "every (or some) algebraic alg-universal category contains a rigid proper class of objects" is again a set-theoretical assumption, the negation of Vopěnka principle (see [12]). In section 4 we present an example of a rigid proper class of accessible set functors. The idea is due to V. Koubek. The following questions naturaly arise:

Open problem 1.1. *Is the category of all accessible set functors and natural transformations universal?*

Open problem 1.2. *Is the (ilegitimate) category of all set functors and natural transformations hyper-universal?*

Notation. We are working in a standard set theory with the axiom of choice (for example ZFC). An ordinal is the set of all smaller ordinals and a cardinal is the least ordinal with its cardinality. Let $f : X \to Y$ be a mapping. $\mathrm{Im}(f)$ denotes the image of f; $f(x)$ means the image of the element $x \in X$; $f[R]$ means the image of the subset $R \subseteq X$; f^{-1} is always the mapping $f^{-1} : PY \to PX$ (where PX is the set of all subsets of X), not the inverse mapping. Let F, G be set functors, $\mu : F \to G$ be a natural transformation. By μ_X we mean the component $\mu_X : FX \to GX$ of μ.

2. Set functors

In this section, we recall some known facts about set functors, which will be needed in this paper. Their proofs can be found in [22, 13]. Every set functor F can be written as a coproduct

$$F = \coprod_{i \in F1} F_i,$$

where all components F_i are connected, i.e. $|F_i 1| = 1$. Each connected set functor either contains precisely one isomorphic copy of the identity functor (this is precisely when it is faithful), or contains precisely one isomorphic copy of the constant functor $\mathbb{C}_1$– the functor which assigns empty set to empty set and a one-point set to all nonempty sets. The following easy criterion will be used:

Proposition 2.1. *Let F be a connected set functor and $x \in FX$ an arbitrary element. Then F is faithful, iff $Ff(x) \neq Fg(x)$ for the two distinct constant mappings $f, g : X \to 2$.*

All set functors in this article are connected and faithful. For this reason, we formulate the next definition and propositions just for this situation. There would be some technical difficulties in the general case. The most important structural properties of a (faithful connected) set functor F are *filters* and *monoids* of elements $x \in FX$.

$$\begin{aligned} \mathrm{Flt}(x) &= \{U \subseteq X \ ; \ (\exists u \in FU) \ Fi(u) = x, \ i : U \to X \text{ is the inclusion } \} \\ &= \{f[U] \ ; \ (\exists u \in FU) \ Ff(u) = x, \ f : U \to X \text{ is a mapping } \} \\ \mathrm{Mon}(x) &= \{f : X \to X \ ; \ Ff(x) = x\}. \end{aligned}$$

Theorem 2.2. *Let F be a faithful connected set functor, $x \in FX$. Then* $\mathrm{Flt}(x)$ *is a filter on X,* $\mathrm{Mon}(x)$ *is a submonoid of the transformation monoid on X and* $\mathrm{Flt}(x) = \{\mathrm{Im}(f) \ ; \ f \in \mathrm{Mon}(x)\}$*. If $U \in \mathrm{Flt}(x)$ and $f \in \mathrm{Mon}(x)$, then $f[U] \in \mathrm{Flt}(x)$.*

F is said to be *κ-accessible*, if for every set X and $x \in FX$ there exists a set $U \in \mathrm{Flt}(x)$ such that $|U| < \kappa$. In the other words, every element can be accessed from an element of an image of some "small" set (small means here, with cardinality less than κ). This definition agrees with the general notion of κ-accessibility (preservation of κ-filtered colimits) from [18]. An ω-accessible functor is called *finitary.*

The category of κ-accessible (κ is a fixed cardinal) set functors and natural transformations is algebraic: A κ-accesible set functor is determined (up to natural equivalence) by its restriction $\mathbf{Card}_{<\kappa} \to \mathbf{Set}$, where $\mathbf{Card}_{<\kappa}$ is the full subcategory of $\mathbf{Set}$ generated by cardinals less than κ. Indeed, the original functor is the Kan extension of this restriction. A functor $G : \mathbf{Card}_{<\kappa} \to \mathbf{Set}$ can be viewed as a many-sorted algebra (sorts are $G\alpha, \alpha < \kappa$) with operations $Gf : G\alpha \to G\beta$ for every $f : \alpha \to \beta$, $\alpha, \beta < \kappa$. Algebra homomorphisms correspond precisely to natural transformations. It is known and easy to see that the category of S-sorted algebras is algebraic for every set S.

The next proposition is easy and often useful.

Proposition 2.3. *Let $\mu : F \to G$ be a natural transformation of faithful connected set functors, X a set, $x \in FX$. Then* $\mathrm{Flt}(x) \subseteq \mathrm{Flt}(\mu_X(x))$, $\mathrm{Mon}(x) \subseteq \mathrm{Mon}(\mu_X(x))$.

Finally, we will need the following simple observation:

Proposition 2.4. *Let $\mu : F \to G$ be a natural transformation of faithful connected set functors, X be a finite set, $x \in FX$. Let $f \in \mathrm{Mon}(x)$ for every bijection $f : X \to X$. Then* $\mathrm{Flt}(\mu_X(x)) = \{X\}$.

Proof. Due to the preceding proposition, we have $f \in \mathrm{Mon}(\mu_X(x))$ for every bijection $f : X \to X$. Suppose, we have $\emptyset \neq U \subset X$, $U \in \mathrm{Flt}(\mu_X(x))$. We can choose a sequence $f_1, \dots, f_n : X \to X$ of bijections, such that $U \cap f_1[U] \cap \dots \cap f_n[U] = \emptyset$. From the last part of 2.2, it follows that $f_i[U] \in \mathrm{Flt}(\mu_X(x))$. Because $\mathrm{Flt}(\mu_X(x))$ is a filter, we have $\emptyset = U \cap f_1[U] \cap \dots \in \mathrm{Flt}(\mu_X(x))$, a contradiction. □

In the situation of this proposition, one can easily see, that $\mathrm{Flt}(x) = \{X\}$ (the same argument as in the proof) and $\mathrm{Mon}(x) = \mathrm{Mon}(\mu_X(x)) = \{f\ ;\ f \text{ is a bijection}\}$ (from 2.2).

3. The full embedding

Theorem 3.1. *The category* $\mathbf{SetFunc_{fin}}$ *of finitary set functors and natural transformations is algebraic and alg-universal.*

Remark. In fact, we will prove a stronger result: The category of 7-accessible connected faithful set functors is alg-universal.

We are going to construct a full embedding $\Phi : \mathbf{Alg}(1,1) \to \mathbf{SetFunc_{fin}}$. This is enough, since the category $\mathbf{Alg}(1,1)$ is alg-universal and $\mathbf{SetFunc_{fin}}$ is algebraic (see sections 1,2).

Let $\mathcal{M} = (M, \alpha, \beta) \in \mathbf{Alg}(1,1)$ be an algebra with two unary operations. For every $m \in M$, we now define a mapping

$$s_{\mathcal{M},m} : P6 \to M \cup \{o, j\}.$$

The union is assumed to be disjoint. For $R \subseteq 6$, we let

$$s_{\mathcal{M},m}(R) = \begin{cases} o & \text{if } R = 0 \\ m & \text{if } |R| = 1 \text{ or } |R| = 5 \\ \alpha(m) & \text{if } |R| = 2 \text{ or } |R| = 4 \\ \beta(m) & \text{if } |R| = 3 \\ j & \text{if } R = 6 \end{cases}$$

Observe, that the mappings $s_{\mathcal{M},m_1}$ and $s_{\mathcal{M},m_2}$ are distinct for distinct $m_1, m_2 \in M$. For a set X and a mapping $f : X \to Y$, we put

$$\begin{aligned} \mathbb{M}X &= \{s_{\mathcal{M},m} g^{-1} : PX \to M \cup \{o, j\};\ m \in M,\ g : 6 \to X \text{ is a map}\} \\ \mathbb{M}f(s_{\mathcal{M},m} g^{-1}) &= s_{\mathcal{M},m} g^{-1} f^{-1}. \end{aligned}$$

$\mathbb{M}$ is a set functor: For every $f_1 : X \to Y, f_2 : Y \to Z$, we have

$$\begin{aligned} \mathbb{M}\mathrm{id}_X(s_{\mathcal{M},m} g^{-1}) &= s_{\mathcal{M},m} g^{-1} \mathrm{id}_X^{-1} = s_{\mathcal{M},m} g^{-1}, \\ \mathbb{M}f_1(\mathbb{M}f_2(s_{\mathcal{M},m} g^{-1})) &= s_{\mathcal{M},m} g^{-1} f_2^{-1} f_1^{-1} = s_{\mathcal{M},m} g^{-1} (f_1 f_2)^{-1} = \\ &= \mathbb{M}f_1 f_2(s_{\mathcal{M},m} g^{-1})). \end{aligned}$$

Let $R \subseteq X$. Let $\chi_{R,X} : X \to 2$ denote the characteristic mapping of R, i.e. $\chi_{R,X}(x) = 1$, iff $x \in R$.

Claim 3.1.1. *The functor* $\mathbb{M}$ *is faithful, connected and 7-accessible.*

Proof. 7-accessibility is clear – every element can be accessed from some $s_{\mathcal{M},m} \in \mathbb{M}6$.

Connectedness: The elements of $\mathbb{M}1$ are of the form $s_{\mathcal{M},m}f^{-1}$, where $f : 6 \to 1$ is the unique mapping. But $s_{\mathcal{M},m}f^{-1}(0) = s_{\mathcal{M},m}(0) = o$ and $s_{\mathcal{M},m}f^{-1}(1) = s_{\mathcal{M},m}(6) = j$, hence $s_{\mathcal{M},m}f^{-1}$ doesn't depend on m – $|\mathbb{M}1| = 1$.

Faithfulness: We will use Proposition 2.1. Take arbitrary $s = s_{\mathcal{M},m} \in \mathbb{M}6$. Then $s\chi_{0,6}^{-1}$ and $s\chi_{6,6}^{-1}$ differs on $\{0\}$:

$$\begin{aligned} s\chi_{0,6}^{-1}(\{0\}) &= s(6) = j \\ s\chi_{6,6}^{-1}(\{0\}) &= s(0) = o \end{aligned}$$

□

Given two algebras $\mathcal{M} = (M, \alpha, \beta)$, $\mathcal{N} = (N, \gamma, \delta)$ and a homomorphism $h : M \to N$, we define a natural transformation $\mu^h : \mathbb{M} \to \mathbb{N}$ as follows

$$\mu_X^h(s_{\mathcal{M},m}g^{-1}) = s_{\mathcal{N},h(m)}g^{-1}.$$

Claim 3.1.2. *The definition is correct.*

Proof. We should check, that if $s_{\mathcal{M},m_1}g_1^{-1} = s_{\mathcal{M},m_2}g_2^{-1}$, then $s_{\mathcal{N},h(m_1)}g_1^{-1} = s_{\mathcal{N},h(m_2)}g_2^{-1}$. For $R \subseteq X$, we have

$$\begin{aligned} s_{\mathcal{N},h(m_1)}g_1^{-1}(R) &= \begin{cases} o & |g_1^{-1}[R]| = 0 \\ h(m_1) & |g_1^{-1}[R]| = 1,5 \\ \gamma(h(m_1)) & |g_1^{-1}[R]| = 2,4 \\ \delta(h(m_1)) & |g_1^{-1}[R]| = 3 \\ j & |g_1^{-1}[R]| = 6 \end{cases} = \\ &= \begin{cases} o & |g_1^{-1}[R]| = \emptyset \\ h(m_1) & |g_1^{-1}[R]| = 1,5 \\ h(\alpha(m_1)) & |g_1^{-1}[R]| = 2,4 \\ h(\beta(m_1)) & |g_1^{-1}[R]| = 3 \\ j & |g_1^{-1}[R]| = 6 \end{cases} = \bar{h}(s_{\mathcal{M},m_1}g_1^{-1}(R)), \end{aligned}$$

where $\bar{h} : M \cup \{o, j\} \to N \cup \{o, j\}$ coincides with h on M and is identical on $\{o, j\}$. The same computation gives $s_{\mathcal{N},h(m_2)}g_2^{-1}(R) = \bar{h}(s_{\mathcal{M},m_2}g_2^{-1}(R))$.
Since $s_{\mathcal{M},m_1}g_1^{-1}(R) = s_{\mathcal{M},m_2}g_2^{-1}(R)$, we are done. □

Claim 3.1.3. *μ is natural.*

Proof. Let $s_{\mathcal{M},m}g^{-1} \in \mathbb{M}X$, $f : X \to Y$ be arbitrary. Then

$$\begin{aligned} \mathbb{N}f(\mu_X^h(s_{\mathcal{M},m}g^{-1})) &= \mathbb{N}f(s_{\mathcal{N},h(m)}g^{-1}) = s_{\mathcal{N},h(m)}g^{-1}f^{-1}, \\ \mu_Y^h(\mathbb{M}f(s_{\mathcal{M},m}g^{-1})) &= \mu_Y^h(s_{\mathcal{N},m}g^{-1}f^{-1}) = s_{\mathcal{N},h(m)}g^{-1}f^{-1}. \end{aligned}$$

□

The functor $\Phi : \mathrm{Alg}(1,1) \to \mathrm{SetFunc}_{\mathrm{fin}}$ given by

$$\Phi(\mathcal{M}) = \mathbb{M}, \quad \Phi(h) = \mu^h$$

is the wanted full and faithful functor.

Claim 3.1.4. *Φ is a faithful functor.*

Proof. It is clear, that Φ preserves the identities and composition.

Faithfulness: Take distinct homomorphisms $h, h' : M \to N$ and then, an element $m \in M$, for which $h(m) \neq h'(m)$. Then $\mu_6^h(s_{\mathcal{M},m}) = s_{\mathcal{N},h(m)} \neq s_{\mathcal{N},h'(m)} = \mu_6^{h'}(s_{\mathcal{M},m})$ from the note after the definition of the mappings $s_{\dots}$. □

Let $\mathcal{M} = (M, \alpha, \beta)$, $\mathcal{N} = (N, \gamma, \delta)$ be algebras. Let $\mu : \mathbb{M} \to \mathbb{N}$ be a natural transformation. We will check that $\mu = \mu^h$ for some homomorphism $h : M \to N$ proving the fullness of Φ.

Claim 3.1.5. *Let $g : 6 \to 6$, $n \in N$. Then $\mathrm{Im}(g) \in \mathrm{Flt}(s_{\mathcal{N},n} g^{-1})$.*

Proof. Take the factorization $g = ih$, where $i : \mathrm{Im}(g) \to 6$ is the inclusion. Then clearly $Fi(s_{\mathcal{N},n} h^{-1}) = s_{\mathcal{N},n} g^{-1}$. □

Claim 3.1.6. *Let $g : 6 \to 6$ be a bijection, then $g \in \mathrm{Mon}(s_{\mathcal{M},m})$.*

Proof. We should check that $s_{\mathcal{M},m}(R) = s_{\mathcal{M},m} g^{-1}(R)$ $(= s_{\mathcal{M},m}(g^{-1}(R)))$ for every $R \subseteq 6$. This is true, since $|g^{-1}(R)| = |R|$ and the value of $s_{\mathcal{M},m}$ on some subset $S \subseteq 6$ depends only on the cardinality of S. □

From these two claims, it follows that the only elements $s \in \mathbb{N}6$ with $\mathrm{Flt}(s) = \{6\}$ are the elements $s_{\mathcal{N},n}$ $(n \in N)$. Combining this with Proposition 2.4, we obtain $\mathrm{Flt}(\mu_6(s_{\mathcal{M},m})) = \{6\}$, hence

$$\mu_6(s_{\mathcal{M},m}) = s_{\mathcal{N},h(m)}$$

for some $h(m) \in N$. Now we aim to show, that this $h : M \to N$ is a homomorphism of the algebras $\mathcal{M}$, $\mathcal{N}$.

Let $d_{\mathcal{M},m} : P2 \to M \cup \{o, j\}$ be the following mapping $(R \subseteq 2)$:

$$d_{\mathcal{M},m}(R) = \begin{cases} o & \text{if } R = 0 \\ m & \text{if } R = \{0\} \text{ or } R = \{1\} \\ j & \text{if } R = 2 \end{cases}$$

Claim 3.1.7. *Let $m \in M$, $R \subseteq 6$. Then*

$$\begin{aligned} d_{\mathcal{M},m} &= \mathbb{M}\chi_{R,6}(s_{\mathcal{M},m}), \text{ if } |R| = 1 \\ d_{\mathcal{M},\alpha(m)} &= \mathbb{M}\chi_{R,6}(s_{\mathcal{M},m}), \text{ if } |R| = 2 \\ d_{\mathcal{M},\beta(m)} &= \mathbb{M}\chi_{R,6}(s_{\mathcal{M},m}), \text{ if } |R| = 3 \end{aligned}$$

In particular $d_{\mathcal{M},m} \in \mathbb{M}2$.

Proof. This is an easy calculation. □

Of course, a similar claim holds for n, γ, δ and the functor $\mathbb{N}$.

Claim 3.1.8. *Let $m \in M$. Then $\mu_2(d_{\mathcal{M},m}) = d_{\mathcal{N},h(m)}$.*

Proof. We use the naturality of μ for $\chi_{R,6} : 6 \to 2$, where $|R| = 1$, and the preceding claim.

$$\begin{aligned}\mathbb{N}\chi_{R,6}(\mu_6(s_{\mathcal{M},m})) &= \mathbb{N}\chi_{R,6}(s_{\mathcal{N},h(m)})) = d_{\mathcal{N},h(m)} \\ &= \mu_2(\mathbb{M}\chi_{R,6}(s_{\mathcal{M},m})) = \mu_2(d_{\mathcal{M},m}).\end{aligned}$$

□

Claim 3.1.9. *Let $m \in M$. Then $h(\alpha(m)) = \gamma(h(m))$.*

Proof. We use the naturality of μ for $\chi_{R,6} : 6 \to 2$, where $|R| = 2$, and the last two claims.

$$\begin{aligned}\mathbb{N}\chi_{R,6}(\mu_6(s_{\mathcal{M},m})) &= \mathbb{N}\chi_{R,6}(s_{\mathcal{N},h(m)})) = d_{\mathcal{N},\gamma(h(m))} \\ &= \mu_2(\mathbb{M}\chi_{R,6}(s_{\mathcal{M},m})) = \mu_2(d_{\mathcal{M},\alpha(m)}) = d_{\mathcal{N},h(\alpha(m))}.\end{aligned}$$

Because the mappings $d_{\mathcal{N},n}, d_{\mathcal{N},n'}$ are distinct for distinct $n, n' \in N$, we have $\gamma(h(m)) = h(\alpha(m))$. □

Claim 3.1.10. *Let $m \in M$. Then $h(\beta(m)) = \delta(h(m))$.*

Proof. The proof is similar to the previous – use a subset $R \subseteq 6$ such that $|R| = 3$. □

We have proved, that h is a homomorphism. To finish the proof, we must observe:

Claim 3.1.11. $\mu = \mu^h$.

Proof. Let $g : 6 \to X$ be an arbitrary mapping, $m \in M$. From the naturality of μ, we have

$$\begin{aligned}\mathbb{N}g(\mu_6(s_{\mathcal{M},m})) &= \mathbb{N}g(s_{\mathcal{N},h(m)}) = s_{\mathcal{N},h(m)}g^{-1} \\ &= \mu_X(\mathbb{M}g(s_{\mathcal{M},m})) = \mu_X(s_{\mathcal{M},m}g^{-1}).\end{aligned}$$

□

4. Rigid proper class of accessible set functors

Let $\mathcal{F}$ be a filter on a set X and $f : X \to Y$ be a mapping. By an f-image of $\mathcal{F}$ is meant the following filter on Y:

$$\begin{aligned}f(\mathcal{F}) &= \{S \subseteq Y \ ; \ f[R] \subseteq S \text{ for some } R \in \mathcal{F}\} \\ &= \{f^{-1}(R) \subseteq Y \ ; \ R \in \mathcal{F}\}\end{aligned}$$

It is known and easy to see that the filter functor $\mathbb{F}$ defined by

$$\begin{aligned}\mathbb{F}X &= \{\mathcal{F} \ ; \ \mathcal{F} \text{ is a filter on } X\} && \text{for a set } X \\ \mathbb{F}f(\mathcal{F}) &= f(\mathcal{F}) && \text{for a mapping } f : X \to Y\end{aligned}$$

is a faithful connected set functor. In this functor $\mathrm{Flt}(\mathcal{F}) = \mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}X$.

For an infinite cardinal κ, we put

$$\mathcal{F}_\kappa = \{R \subseteq \kappa \ ; \ |\kappa - R| < \kappa\}.$$

It is easy to see that $\mathcal{F}_\kappa$ is a filter on κ.

Let A be a nonempty class of regular cardinals. For a set X and a mapping $f : X \to Y$ we define

$$\begin{aligned} \mathbb{A}X &= \{g(\mathcal{F}_\kappa) \ ; \ \kappa \in A, \ g : \kappa \to X\} \\ \mathbb{A}f(g(\mathcal{F}_\kappa)) &= fg(\mathcal{F}_\kappa) \end{aligned}$$

$\mathbb{A}$ is a subfunctor of the filter functor $\mathbb{F}$. Hence it is faithful and connected and $\mathrm{Flt}(\mathcal{F}) = \mathcal{F}$ for every $\mathcal{F} \in \mathbb{A}X$. It is λ-accessible for every cardinal λ greater than all $\kappa \in A$.

Theorem 4.1. *Let A, B be nonempty classes of regular cardinals. Then there exists a natural transformation $\mathbb{A} \to \mathbb{B}$, iff $A \subseteq B$. In this case, it is unique.*

Proof. First, we describe the filters $f(\mathcal{F}_\kappa)$ for a regular cardinal κ and $f : \kappa \to X$. Let $U \subseteq V \subseteq X$. Let $\mathcal{F}_{U,V,X,\kappa}$ be the following filter on X:

$$\mathcal{F}_{U,V,X,\kappa} = \{R \subseteq X \ ; \ U \subseteq R, \ |V - R| < \kappa\}$$

Note that

- If $U, U' \subseteq X$, $U \neq U'$, then $\mathcal{F}_{U,V,X,\kappa} \neq \mathcal{F}_{U',V',X,\lambda}$ for every V, V', where $U \subseteq V \subseteq X$, $U' \subseteq V' \subseteq X$, and κ, λ are regular cardinals.
- Let $V, V' \subseteq \kappa$, $|V| = \lambda$. Then $\mathcal{F}_{0,V,\kappa,\lambda} = \mathcal{F}_{0,V',\kappa,\lambda}$ iff the symmetric difference $(V - V') \cup (V' - V)$ has cardinality less than λ.

Claim 4.1.1. *Let κ be a regular cardinal, $f : \kappa \to X$ be a mapping. Let $U = \{x \ ; \ |f^{-1}(\{x\})| = \kappa\}$, $V = f[\kappa]$. Then $f(\mathcal{F}_\kappa) = \mathcal{F}_{U,V,X,\kappa}$. If $U = 0$ then $|V| = \kappa$.*

Proof. The inclusion "$\subseteq$". Let $R \in f(\mathcal{F}_\kappa)$, so $|\kappa - f^{-1}(R)| < \kappa$. If $x \in U$ and $x \notin R$, then $|\kappa - f^{-1}(R)| \geq |\kappa - f^{-1}(X - \{x\})| = |f^{-1}(\{x\})| = \kappa$, a contradiction, hence $U \subseteq R$. Since moreover $|f[\kappa] - R| \leq |\kappa - f^{-1}(R)| < \kappa$, we have $R \in \mathcal{F}_{U,V,X,\kappa}$.

The inclusion "$\supseteq$". Let $R \in \mathcal{F}_{U,V,X,\kappa}$, so $U \subseteq R$, $|V - R| < \kappa$. Since $\kappa - f^{-1}(R) = \cup_{x \in V-R} f^{-1}(\{x\})$, we have $|\kappa - f^{-1}(R)| < \kappa$ (the right hand side is a union of less then κ sets, each of cardinality fewer than κ, κ is regular). Thus $R \in f(\mathcal{F}_\kappa)$.

The last statement is obvious. □

Now, let $\mu : \mathbb{A} \to \mathbb{B}$ be a natural transformation.

Claim 4.1.2. *Let $\kappa \in A$. Then $\kappa \in B$ and $\mu_\kappa(\mathcal{F}_\kappa) = \mathcal{F}_\kappa$.*

Proof. Let $\lambda \in B$, $f : \lambda \to \kappa$, $U \subseteq V \subseteq \kappa$ be such that $\mu_\kappa(\mathcal{F}_\kappa) = f(\mathcal{F}_\lambda) = \mathcal{F}_{U,V,\kappa,\lambda}$.

Every bijection is in the monoid of $\mathcal{F}_\kappa \in \mathbb{A}\kappa$. According to 2.4, every bijection is in the monoid of $\mathcal{F}_{U,V,\kappa,\lambda}$. It is obvious that $b(\mathcal{F}_{U,V,\kappa,\lambda}) = \mathcal{F}_{b[U],b[V],\kappa,\lambda}$. Thus $b[U] = U$ for every bijection (see the note above), hence either $U = 0$ or $U = \kappa$.

Suppose $U = \kappa$. Let $x \in \kappa$ be arbitrary. The set $X - \{x\}$ is in the filter of $\mathcal{F}_\kappa$, but it isn't in the filter of $\mathcal{F}_{\kappa,\kappa,\kappa,\lambda}$. This contradicts 2.4 (recall that $\mathrm{Flt}(\mathcal{F}) = \mathcal{F}$).

Now, we have $U = 0$, thus $\lambda = |V|$ (see the last statement in the previous claim). If $|\kappa - V| = \kappa$, we can find a bijection such that the symmetric difference $(V - b[V]) \cup (b[V] - V)$ has cardinality κ, hence $\mathcal{F}_{0,V,\kappa,\lambda} \neq \mathcal{F}_{0,b[V],\kappa,\lambda}$ (see the note above again), a contradiction. Hence $\lambda = \kappa$ and $|\kappa - V| < \kappa$. Then $\mathcal{F}_{0,V,\kappa,\kappa} = \mathcal{F}_{0,\kappa,\kappa,\kappa} = \mathcal{F}_\kappa$. □

We now know that $A \subseteq B$ and $\mu_\kappa(\mathcal{F}_\kappa) = \mathcal{F}_\kappa$. From the naturality of μ, it follows that for every $\kappa \in A$, set X and mapping $f : \kappa \to X$

$$\mu_X(f(\mathcal{F}_\kappa)) = f(\mu_\kappa(\mathcal{F}_\kappa)) = f(\mathcal{F}_\kappa).$$

Thus the transformation μ is uniqely determined - it is the inclusion. □

Let $\mathcal{A}$ be a conglomerate (i.e. collection of classes in the sense of [1]) of pairwise incomparable classes of regular cardinals. From the last theorem, it follows that $\{\mathbb{A} \ ; \ A \in \mathcal{A}\}$ is a rigid conglomerate of set functors. Putting $\mathcal{A} = \{\{\kappa\} \ ; \ \kappa \text{ is a regular cardinal}\}$, we obtain:

Corollary 4.2. *There exists a rigid proper class of accessible set functors.*

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Mathematical Institute of Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic
E-mail address: barto@karlin.mff.cuni.cz