Baskets of essentially algebraic categories

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Concrete categories

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K-morphisms $(H, \mathbf{K} - \text{str}) \to (H', \mathbf{K} - \text{str'})$:
Some $\mathbf{H}$-morphisms $H \to H'$

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$U(H, \mathbf{K} - \text{str}) = H$

Slices

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there exist functors $\Phi : \mathbf{K} \to \mathbf{K}'$, $F : \mathbf{H} \to \mathbf{H}'$ such that

- $U'\Phi = FU$ and
- for every $\mathbf{K}$-objects $K = (H, \dots), L = (J, \dots)$ and $\mathbf{H}$-morphism $f : H \to J$

$$f : K = (H, \dots) \to L = (J, \dots) \text{ is a } \mathbf{K}\text{-morphism}$$

iff

$$Ff : \Phi K = (FH, \dots) \to \Phi L = (FJ, \dots) \text{ is a } \mathbf{K}'\text{-morphism}$$

Baskets

FACT The relation "slice" is a quasi-ordering (reflexive and transitive).

We can form an equivalence

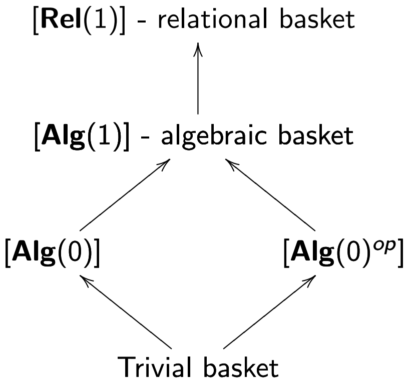
$$U \sim_{slice} U'$$

iff

U is a slice of U' and vice versa

Equivalence "classes" are called baskets (of concrete categories).

Basic baskets



The category **Fix**(2)

Objects: (A, α_0, α_1)

- A is a set
- α_0 is a (total) unary operation $\alpha_0 : A \to A$
- α_1 is a partial unary operation $\alpha_1 : \mathrm{Def}(\alpha_1) \to A$
- $\mathrm{Def}(\alpha_1) = \{a \mid \alpha_0(a) = a\}$

Morphisms: Homomorphisms of partial algebras.

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This is an example of essentially algebraic category of height 2.

The category **Fix**(3)

Objects: $(A, \alpha_0, \alpha_1, \alpha_2)$

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- α_1 is a partial unary operation $\alpha_1 : \mathrm{Def}(\alpha_1) \to A$
- $\mathrm{Def}(\alpha_1) = \{a \mid \alpha_0(a) = a\}$
- α_1 is a partial unary operation $\alpha_1 : \mathrm{Def}(\alpha_1) \to A$
- $\mathrm{Def}(\alpha_2) = \{a \mid \alpha_0(a) = a,\ \alpha_1(a) = a\}$

Morphisms: Homomorphisms of partial algebras.

This is an example of essentially algebraic category of height 3.

Essentially algebraic categories

"DEFINITION" Essentially algebraic theory of height α:

- A set of operational symbols, each operational symbol has its level $< \alpha$
- A set of identities
- For every operational symbol σ, a set of identities $\mathrm{Def}(\sigma)$ in operational symbols of smaller level than the level of σ

Possibly many-sorted, infinitary

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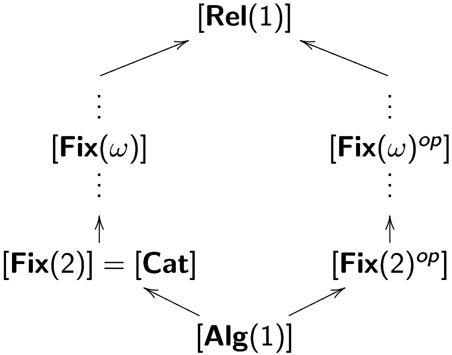
EXAMPLE The category of small categories is essentially algebraic category of height 2.

Baskets of essentially algebraic categories



A theorem and a problem

THEOREM L. B. 06 Every essentially algebraic category of height α is a slice of $\mathbf{Fix}(\alpha)$.

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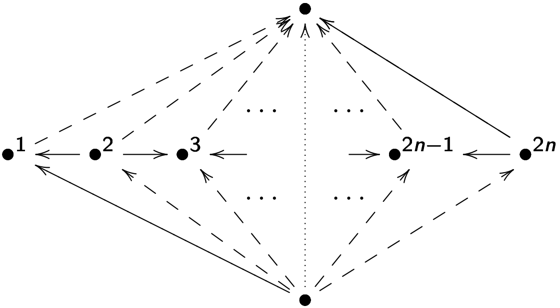
OPEN PROBLEM Find all baskets of essentially algebraic categories.

Why baskets differ?

Every slice $U : \mathbf{K} \to \mathbf{H}$ of $\mathbf{Alg}(1)$ satisfies:



(zz^1)

Slices of **Alg**(1)

THEOREM J. Sichler, V. Trnková 91 Let **H** be a small category. Then $U : \mathbf{K} \to \mathbf{H}$ is a slice of **Alg**(1), iff U satisfies (zz^1).

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Slices of $\mathbf{Alg}(1)$

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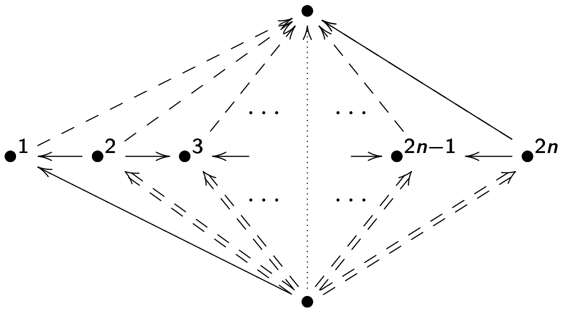
There is a generalization L. B. 06.

Multiple zig-zags

Every slice of **Fix**(2) satisfies



$$(zz^2)$$

Slices of **Fix**(α)

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OPEN PROBLEM Prove this theorem for arbitrary **H** (or at least for **H** $=$ **Set**).

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Thank you for your attention!