The category of varieties and interpretations is alg-universal

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Abstract

A category is said to be alg-universal, if every category of universal algebras can be fully embedded into it. We prove here that the category of varieties and interpretations, or in other words, the category of abstract clones and clone homomorphisms, is alg-universal.

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1 Introduction

The lattice $\mathcal{L}$ of interpretability types of varieties (of finitary monosorted universal algebras) was first introduced and investigated in [1]. Then an issue [2] of Memoirs of the AMS was devoted to the study of $\mathcal{L}$. One of the many open problems formulated there, whether the breadth of this lattice is uncountable, was solved in [3]. The authors proved there (among other) that every poset can be embedded into $\mathcal{L}$ and that the existence of proper class antichain is equivalent to the negation of Vopěnka's principle (see [4]).

In fact, they investigated the category **Clone** of all abstract clones and all their homomorphisms and then used the well-known fact that $\mathcal{L}$ can be obtained by forming a partially ordered class from the category **Clone** in a standard way

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(we introduce a quasiordering on objects – $A \leq B$ iff $\mathbf{Clone}(A, B) \neq \emptyset$ and then make a partial ordering from $\leq$). They constructed a semifull embedding from the category of semigroups to **Clone**, i.e. a functor $\Phi : \mathbf{Smg} \to \mathbf{Clone}$ such that $\mathbf{Smg}(A, B) \neq \emptyset$ precisely when $\mathbf{Clone}(\Phi A, \Phi B) \neq \emptyset$, for every $A, B \in \mathrm{Obj}(\mathbf{Smg})$. The mentioned results are consequences of the fact that the category of semigroups is alg-universal, i.e. every category of universal algebras can be fully embedded into it. See Section 2 for more information about alg-universal categories with the corresponding references. In the same article, the authors also proved that every group is isomorphic to the endomorphism monoid of some clone A.

Here we prove a substantial strengthening of both results by answering the open problem formulated there – the category **Clone** is alg-universal. Moreover, our construction uses idempotent clones only, while the constructions in [3] use many constant operations. To state the main result, let us use an alternative formulation (see the next paragraph):

The category of idempotent varieties and interpretations is alg-universal.

There are several ways how to view a variety: class of algebras, equational theory, finitary monad over **Set** or clone (the last two describe variety up to term equivalence). Clone homomorphisms then correspond to concrete functors (going in the opposite direction), interpretations and monad homomorphisms respectively. We recall these well-known facts in Section 3.

Some basic notions and results from the theory of rewriting systems, which we will need for the proof, are recalled in Section 4.

Section 5 contains the proof of the main theorem. To enhance readability, several facts are formulated there, their proofs are in Sections 6, 7, 8.

2 Alg-universal categories

Recall that a functor $\Phi : \mathbf{K} \to \mathbf{L}$ is a *full-embedding*, if it is bijective on hom-sets. A category $\mathbf{K}$ is said to be *alg-universal*, if every category $\mathbf{Alg}(\Sigma)$ of algebras with the signature Σ can be fully embedded into it, or in other words, if it contains an isomorphic copy of $\mathbf{Alg}(\Sigma)$ for every Σ. A category $\mathbf{K}$ is said to be *algebraic*, if it can be fully embedded into $\mathbf{Alg}(\Sigma)$ for some signature Σ. We note that there are many definitions of the term "algebraic" in the literature, this definition is used in the theory of representations in categories.

Surprisingly many algebraic categories turned out to be alg-universal, e. g. the category **Rel**(2) of graphs and graph homomorphisms, the category $\mathbf{Alg}(1,1)$ of algebras with two unary operations and algebra homomorphisms (both in [5]), the variety of $(0,1)$-lattices [6], semigroups [7], integral domains of characteristic zero [8], and many others. These older results are summarized in the monograph [9] and in the survey article [10], where many later results are also mentioned, e.g. the full characterization of alg-universal varieties of $(0,1)$-lattices [11] and of semigroups [12].

Recently, the category of finitary endofunctors of **Set** and natural transformations was proved to be alg-universal [13]. The idea of this construction lead the author to the proof of the main theorem.

Recall the following properties of any algebraic alg-universal category **K**:

- Every category with set many objects can be fully embedded into **K**. In particular, every monoid can be represented as the monoid of all endomorphisms of some **K**-object (see [9]).
- Recall that a class of **K**-objects is called *rigid*, if there are no morphisms between them (except the identity automorphisms). From the last item, it follows that **K** contains a rigid set of arbitrary cardinality. The statement "There exists a rigid proper class of **K**-objects" is equivalent to the negation of set-theoretical Vopěnka's principle (see [4], [10]).
- The statement "Every concretizable category (i.e. a category which admits a faithful functor to **Set**) can be fully embedded into **K**." is equivalent to "The class of measurable cardinals is a set" (see [9], the condition (M)).

To prove that a certain category is alg-universal, it suffices to fully embed any alg-universal category into it. Our "testing" category for the main result is the following auxiliary category.

Definition 1 $\mathbf{Alg}_*(1,1)$ *is the full subcategory of* $\mathbf{Alg}(1,1)$ *consisting of algebras* (A,α,β) *such that* $a,\alpha(a),\beta(a)$ *are pairwise distinct for every* $a \in A$.

Proposition 2 $\mathbf{Alg}_*(1,1)$ *is alg-universal.*

PROOF. We will construct a full embedding $\Phi : \mathbf{Alg}(1,1) \to \mathbf{Alg}(1,1)$ such that for every $\mathcal{A} = (A,\alpha,\beta) \in \mathbf{Alg}(1,1)$, the algebra $\Phi(\mathcal{A}) = (\overline{A},\overline{\alpha},\overline{\beta})$ will satisfy $\overline{\alpha}(\overline{a}) \neq \overline{a}$, $\overline{\alpha}(\overline{a}) \neq \overline{\beta}(\overline{a})$ for all $\overline{a} \in \overline{A}$. Moreover, if $\alpha(a) \neq a$ for every $a \in A$, then $\overline{\beta}(\overline{a}) \neq \overline{a}$ for every $\overline{a} \in \overline{A}$. Therefore $\Phi\Phi$ will be a full embedding $\mathbf{Alg}(1,1) \to \mathbf{Alg}_*(1,1)$.

For an algebra $\mathcal{A} = (A,\alpha,\beta)$, let $\Phi(\mathcal{A}) = (\overline{A},\overline{\alpha},\overline{\beta})$ be as follows:

$$\overline{A} = 3 \cup A \times 2$$

$$\begin{aligned}
\overline{\alpha}(0) &= 1, \quad \overline{\alpha}(1) = 0, \quad \overline{\alpha}(2) = 1 \\
\overline{\alpha}(a,0) &= 2 \\
\overline{\alpha}(a,1) &= (\beta(a),0) \\
\overline{\beta}(0) &= 2, \quad \overline{\beta}(1) = 2, \quad \overline{\beta}(2) = 0 \\
\overline{\beta}(a,0) &= (a,1) \\
\overline{\beta}(a,1) &= (\alpha(a),1).
\end{aligned}$$

For a homomorphism $f : (A, \alpha, \beta) \to (B, \gamma, \delta)$, let

$$\Phi(f) = \overline{f} = id_3 \cup f \times 2.$$

It is easy to see, that Φ is a faithful functor and that $(\overline{A}, \overline{\alpha}, \overline{\beta})$ has all required properties. It remains to prove that Φ is full. So, let $g : (\overline{A}, \overline{\alpha}, \overline{\beta}) \to (\overline{B}, \overline{\gamma}, \overline{\delta})$ be a homomorphism. We have to prove that $g = \overline{f}$ for some homomorphism $f : A \to B$.

1. Observe that $\overline{\alpha}(\overline{\alpha}(0)) = 0$ and the only elements $\overline{b} \in \overline{B}$ for which $\overline{\gamma}(\overline{\gamma}(\overline{b})) = \overline{b}$ are $0, 1$. Hence $g(0) \in \{0, 1\}$, since g is a homomorphism.

2. Suppose $g(0) = 1$. Then $g(1) = 0$ (because $g(1) = g(\overline{\alpha}(0)) = \overline{\gamma}(g(0)) = 0$), $g(2) = 2$ (because $g(2) = g(\overline{\beta}(1)) = \overline{\delta}(g(1)) = 2$). But $0 = g(\overline{\alpha}(2)) = \overline{\gamma}(g(2)) = 1$, a contradiction.

3. We have $g(0) = 0$, thus $g(1) = 1$ (because $g(1) = g(\overline{\alpha}(0)) = \overline{\gamma}(g(0)) = 1$) and $g(2) = 2$ (because $g(2) = g(\overline{\beta}(0)) = \overline{\delta}(g(0)) = 2$)).

4. For every $a \in M$, we have $2 = g(2) = g(\overline{\alpha}(a,0)) = \overline{\gamma}(g(a,0))$. The only elements of $\overline{B}$ which are sent to 2 by $\overline{\gamma}$ are the elements $(b, 0)$. Therefore $g(a,0) = (f(a),0)$ for some mapping $f : M \to N$. Moreover $g(a,1) = g(\overline{\beta}(a,0)) = \overline{\delta}(g(a,0)) = \overline{\delta}(f(a),0) = (f(a),1)$.

5. Now, we have $g = \overline{f}$. It remains to prove that f is a homomorphism: $(f(\beta(a)),0) = g(\overline{\alpha}(a,1)) = \overline{\gamma}(g(a,1)) = \overline{\gamma}(f(a),1) = (\delta(f(a)),0)$, and $(f(\alpha(a)),1) = g(\overline{\beta}(a,1)) = \overline{\delta}(g(a,1)) = \overline{\delta}(f(a),1) = (\gamma(f(a)),1)$. This concludes the proof. $\square$

Remark 3 *The referee suggested another proof of Proposition 2. We briefly describe his construction: We define a full embedding Φ from the alg-universal category of directed graphs (X, R) without loops satisfying $xR \neq \emptyset$, $Rx \neq \emptyset$ for every $x \in X$ (see [9]) into* $\mathbf{Alg}_*(1,1)$*: Set $\Phi(X,R) = (F(X) \cup R, \alpha, \beta)$, where $(F(X), \alpha|F(X), \beta|F(X))$ is the free algebra in* $\mathbf{Alg}(1,1)$ *generated by X, $\alpha(x,x') = x$, $\beta(x,x') = x'$ for every $(x,x') \in R$ (and set $\Phi(f) = F(f) \cup f^2$ for morphisms). It can be easily checked that Φ is a faithful functor. Its fullness* *follows from the fact that $X \subseteq \Phi(X,R)$ is the intersection of the images of α and β.*

3 Varieties, interpretations

The basic notions such as universal algebras, varieties, terms, etc. are used in the standard way, see e. g. [14], [15]. We recall several notions to fix the notation.

A (finitary) *signature* is a set Σ of operational symbols together with a mapping arity : $\Sigma \to \omega$. To avoid some technical difficulties, we assume that there is no nullary operation in any signature. All signatures in this paper have this property.

Let $\mathbb{V}$ be a (monosorted) variety of a (finitary) signature Σ. Let X be a fixed countably infinite set. In this paper, we assume that $\{x, y, x_0, \dots, x_{18}\} \subset X$. The absolutely free algebra on X in the signature Σ (the algebra of terms in the operational symbols in Σ over the set X) will be denoted by $\mathrm{Term}(\Sigma)$. An endomorphism of $\mathrm{Term}(\Sigma)$ is called a *substitution*, it is determined by values on variables.

The *equational theory* of $\mathbb{V}$, i.e. the fully invariant congruence of $\mathrm{Term}(\Sigma)$ determined by $\mathbb{V}$, will be denoted $\approx_{\mathbb{V}}$. The congruence $\approx_{\mathbb{V}}$ is often given by its generating set – *base*.

$\mathbb{V}$ is said to be *idempotent*, if $\sigma(x, \dots, x) \approx_{\mathbb{V}} x$ for all $\sigma \in \Sigma$ or, equivalently, for all $\sigma \in \mathrm{Term}(\Sigma)$.

An (abstract) *clone*, in its algebraic definition, is an ω-sorted algebra (C_n, S^n_m, e^n_i) with underlying sets C_n for $n \in \omega$, constants $e^n_i \in C_n$ for $i < n \in \omega$ and heterogeneous operations $S^n_m : C_n \times (C_m)^n \to C_m$, where the following identities hold:

(i) $S^n_k(u; S^m_k(v_1; w_1, \dots, w_m), \dots, S^m_k(v_n; w_1, \dots, w_m)) =$
$= S^m_k(S^n_m(u; v_1, \dots, v_n); w_1, \dots, w_m)$,

(ii) $S^n_n(u; e^n_0, \dots, e^n_{n-1}) = u$,

(iii) $S^n_m(e^n_i; v_0, \dots, v_{n-1}) = v_i$

for any $m, n, k \in \omega$, $u \in C_n$, $v_1, \dots, v_n \in C_m$ and $w_1, \dots, w_m \in C_k$. *Clone homomorphism* $f : (C_n, S^n_m, e^n_i) \to (C'_n, S'^n_m, e'^n_i)$ is a homomorphism of this heterogeneous algebras – a family of mappings $f = \{f_n : C_n \to C'_n \mid n \in \omega\}$ respecting the operations.

From the variety $\mathbb{V}$ we can form its clone $\mathrm{Clone}(\mathbb{V})$ by putting C_n to be the free algebra on the set $\{e_0^n, \ldots, e_{n-1}^n\}$ and $S_m^n(u; v_0, \ldots, v_{n-1})$ to be the image of u under the homomorphism $C_n \to C_m$ which takes each e_i^n to v_i. Conversely, every clone is a clone of "many" varieties which have the same variety of termal operations (see [16]).

Let $\mathbb{V}, \mathbb{W}$ be varieties of signatures Σ, Γ respectively. By an *interpretation* of $\mathbb{V}$ in $\mathbb{W}$, we mean a mapping $\nu : \mathrm{Term}(\Sigma) \to \mathrm{Term}(\Gamma)$ such that

(i) $\nu(x) = x$ for every $x \in X$. If $t \in \mathrm{Term}(\Sigma)$ is a term over $Y \subseteq X$, then $\nu(t)$ is a term over Y.
(ii) ν preserves substitutions, i.e. $\nu(t(s_0, \ldots, s_n)) = \nu(t)(\nu(s_0), \ldots, \nu(s_n))$ if the left hand side is defined.
(iii) ν preserves equations, i.e. if $s \approx_{\mathbb{V}} t$, then $\nu(s) \approx_{\mathbb{W}} \nu(t)$.

We identify ν and ν', if $\nu(s) \approx_{\mathbb{W}} \nu'(s)$ for all $s \in \mathrm{Term}(\Sigma)$. More precisely, an interpretation should be defined as a mapping $\nu : \mathrm{Term}(\Sigma) \to \mathrm{Term}(\Gamma)/\approx_{\mathbb{W}}$.

It is clear that ν is determined by values on the terms $\sigma(x_0, \ldots, x_n)$, $\sigma \in \Sigma$ and that in (iii) it suffices to consider only equations from some base of $\approx_{\mathbb{V}}$.

An interpretation $\nu : \mathrm{Term}(\Sigma) \to \mathrm{Term}(\Gamma)$ determines a clone homomorphism $\mathrm{Clone}(\mathbb{V}) \to \mathrm{Clone}(\mathbb{W})$ and vice versa, see [16].

We can also form a concrete functor (i.e. a functor which commutes with the forgetful functors) $\mathbb{W} \to \mathbb{V}$ from an interpretation in a natural way, and vice versa, see [17].

Finally, interpretations between varieties precisely correspond to monad homomorphisms between their monads. For these notions and related results, we refer to [18].

Altogether, the following categories are equivalent.

(i) The category of varieties and interpretations.
(ii) The dual of the category of varieties and concrete functors.
(iii) The category of abstract clones and clone homomorphisms.
(iv) The category of finitary monads over **Set** and monad homomorphisms.

Remark 4 *Strictly speaking, (i) and (ii) are not correct formulations, because a variety is a* class *of algebras. But this can be obviously avoided.*

4 Terms, rewriting systems

Here we recall some notions and results about terms and term rewriting systems, see [19] for their proofs.

Let Σ be a signature.

A term t over X (in the signature Σ) can be viewed as a labeled tree, where *leaves* are labeled by elements of X, nodes are labeled by elements of $\sigma \in \Sigma$ and every node labeled by σ has arity(σ) sons.

A *height* ht(t) of a term t has its obvious meaning, we should just mention that height of a variable is 0.

By an *address* we mean a finite (possibly empty) sequence of natural numbers 0,1,.... By a subterm of t at the address A, we mean the term $t[A]$ defined inductively by

1. $t[\emptyset] = t$.
2. If $A = Bi$, $t[B] = \sigma(t_0, t_1, \ldots, t_n)$ and $i \leq n$, then $t[A] = t_i$; otherwise $t[A]$ is undefined.

If $t[A]$ is defined, we say that A is a *valid* address in t. We say that s is a *subterm* of t and write $s \subseteq t$, if $s = t[A]$ for some valid address A.

An *equation* (E) (called also *rewriting rule* in some situations) is a pair of terms $(E) = (u, v)$ often written in the form $(E) = u \approx v$.

We say that a term s can be rewritten in one step to t using (E) and write $s \xrightarrow{(E)}_1 t$, if there exists a valid address A in s and a substitution f such that $s[A] = f(u)$ and t is obtained by replacing the subterm $f(u)$ by $f(v)$ at A. We can also say that (E) can be applied to s at the address A and t is the result of the application.

Note that if we consider an equation $u \approx v$ as a rewriting rule, the ordering of the pair is important – $v \approx u$ is another rewriting rule. Rewriting rules are read from left to right.

Let $\mathcal{S}$ be a set of equations (called also *rewriting system*) and $\approx_{\mathcal{S}}$ denote the equational theory it generates. We write $s \xrightarrow{\mathcal{S}}_n t$, if $s = r_0 \xrightarrow{(S_1)}_1 r_1 \ldots \xrightarrow{(S_n)}_1 r_n = t$ for some $(S_i) \in \mathcal{S}$, and write $s \xrightarrow{\mathcal{S}} t$ (and say that s can be rewritten to t), if $s \xrightarrow{\mathcal{S}}_n t$ for some n. A term t is called *reduced*, if no rewriting rule from $\mathcal{S}$ can be applied to t.

It is known and easy to see that $s \approx_{\mathcal{S}} t$, iff there exists a sequence $r_0, \ldots, r_n$

of terms such that $s = r_0 \stackrel{\mathcal{S}}{\longrightarrow}_1 r_1 \stackrel{\mathcal{S}}{\longleftarrow}_1 r_2 \longrightarrow \dots r_n = t$. Such a sequence is called a *derivation* of $s \approx_{\mathcal{S}} t$.

$\mathcal{S}$ is said to be *finitely terminating*, if every sequence of the form $t_0 \stackrel{\mathcal{S}}{\longrightarrow}_1 t_1 \stackrel{\mathcal{S}}{\longrightarrow}_1 t_2 \dots$ is finite. It is said to be *confluent* (resp. *locally confluent*), if for arbitrary terms t, s_0, s_1 such that $t \stackrel{\mathcal{S}}{\longrightarrow} s_0, s_1$ (resp. $t \stackrel{\mathcal{S}}{\longrightarrow}_1 s_0, s_1$), there exists a term r such that $s_0, s_1 \stackrel{\mathcal{S}}{\longrightarrow} r$. If S is finitely terminating and locally confluent, then it is confluent. In this situation, every term s can be rewritten to a unique reduced term $\mathrm{Red}_{\mathcal{S}}(s)$ called *reduced form* of s. Moreover $s \approx_{\mathcal{S}} t$ iff $\mathrm{Red}_{\mathcal{S}}(s) = \mathrm{Red}_{\mathcal{S}}(t)$.

In order to verify that $\mathcal{S}$ is locally confluent, we need not prove $s_0, s_1 \stackrel{\mathcal{S}}{\longrightarrow} r$ for all $t \stackrel{\mathcal{S}}{\longrightarrow}_1 s_0, s_1$. It is enough to consider *critical overlaps* (see [19], pp 134-141), i.e. we can assume that s_0 is the result of the application of $(E_0) = u \approx v \in \mathcal{S}$ at $\emptyset$ (hence $t = f(u)$ for some substitution f) and s_1 is the result of the application of $(E_1) \in \mathcal{S}$ at A, where A is a valid address of u and not an address of some leaf of u.

By a *reduced height* of a term s is meant the height of the reduced form of s.

5 Main Theorem

Theorem 5 *The category* **IdempVar** *of idempotent varieties and interpretations is alg-universal.*

Remark 6 *It is easy to see that* **IdempVar** *is algebraic (see [3], for example).*

As mentioned, we are going to construct a full embedding $\Phi : \mathbf{Alg}_*(1,1) \to$ **IdempVar**. This is sufficient due to Proposition 2.

For an algebra $\mathcal{A} = (A, \alpha, \beta) \in \mathbf{Alg}_*(1,1)$, let $\Sigma_{\mathcal{A}}$ be the signature consisting of 19-ary operational symbols c_a, $a \in A$ and binary operational symbols d_a, $a \in A$. Let $\mathbb{A}$ be the variety whose equational theory is based on

- (C) $c_a(x_0, x_1, \dots, x_{18}) \approx c_a(x_{\sigma(0)}, x_{\sigma(1)}, \dots, x_{\sigma(18)})$ for every permutation σ on 19,
- $(D1)$ $c_a(x, 18y) \approx d_a(x, y)$,
- $(D3)$ $c_a(3x, 16y) \approx d_{\alpha(a)}(x, y)$,
- $(D7)$ $c_a(7x, 12y) \approx d_{\beta(a)}(x, y)$,
- $(E0)$ $d_a(d_a(x, y), y) \approx d_a(x, y)$,
- $(E1)$ $d_a(x, d_a(x, y)) \approx d_a(x, y)$,
- (I) $d_a(x, x) \approx x$.

Each row is to be understood as a set of equations, for example (C) says that for every $a \in A$ and every permutation on 19, we have the equation $c_a(x_0, x_1, \dots, x_{18}) \approx c_a(x_{\sigma(0)}, x_{\sigma(1)}, \dots, x_{\sigma(18)})$. In $(D1), (D3), (D7)$ we use the following abbreviation: $c_a(3x, 16y)$ denotes any term of the form $c_a(W)$, where there are 3 occurrences of x and 16 occurrences of y in W, for example the term $c_a(y, y, x, y, x, y, x, y, y, \dots, y)$.

For a homomorphism $f : (A, \alpha, \beta) \to (B, \gamma, \delta)$ we define an interpretation $\nu_f : \mathrm{Term}(\Sigma_{\mathcal{A}}) \to \mathrm{Term}(\Sigma_{\mathcal{B}})$ of $\mathbb{A}$ in $\mathbb{B}$ by

$$\nu_f(d_a(x, y)) = d_{f(a)}(x, y), \quad \nu_f(c_a(x_0, \dots, x_{18})) = c_{f(a)}(x_0, \dots, x_{18}).$$

The functor $\Phi : \mathbf{Alg}_*(1,1) \to \mathbf{IdempVar}$ defined by $\Phi(\mathcal{A}) = \mathbb{A}$ on objects and by $\Phi(f) = \nu_f$ on morphisms is the proposed full and faithful functor.

We postpone the proof of the following facts after the proof of the theorem. For the following facts, let $\mathcal{A} \in \mathbf{Alg}_*(1,1)$ and $\mathbb{A} = \Phi(\mathcal{A})$.

Fact 1. The rewriting rules $(D1), (D3), (D7), (E0), (E1), (I)$ form a finitely terminating confluent rewriting system (read the rewriting rules from left to right, see also Section 4). For any terms s, t in $\Sigma_{\mathcal{A}}$, we have $s \approx_{\mathbb{A}} t$ iff $\mathrm{Red}(s) \sim_{\mathbb{A}} \mathrm{Red}(t)$, where $\sim_{\mathbb{A}}$ is the equational theory based on (C) and $\mathrm{Red}(s)$ is the reduced form of s in the rewriting system $(D1), (D3), (D7), (E0), (E1), (I)$ (we omit the subscript of Red).

From now on by "reduced, reduced height, …", we mean reduced, reduced height with respect to the above rewriting system. It is clear that if $t \sim_{\mathbb{A}} s$ and t is reduced, then s is also reduced.

Fact 2. Let t be a term over $\{x, y\}$ in $\Sigma_{\mathcal{A}}$ such that $t(t(x, y), y) \approx_{\mathbb{A}} t(x, y)$, $t(x, t(x, y)) \approx_{\mathbb{A}} t(x, y)$. Then t is of reduced height at most 1.

Fact 3. Let $\mathcal{P} = \{1, 3, 7, 12, 16, 18\}$, $P \subseteq \{x_0, x_1, \dots, x_{18}\}$, $|P| \in \mathcal{P}$. The substitution g_P sending all variables in P to x and all other variables to y is called *permissible substitution.* Let t be a term over $\{x_0, \dots, x_{18}\}$ in $\Sigma_{\mathcal{A}}$ such that $g_P(t)$ is of reduced height at most 1 for every permissible substitution g_P. Then the reduced height of t is at most 1.

First, observe that Φ is a correctly defined faithful functor. For better readability, we write $\nu(d_a)$ instead of $\nu(d_a(x, y))$, $\nu(c_a)$ instead of $\nu(c_a(x_0, \dots, x_{18}))$, and so on.

1. For every $\mathcal{A}$, $\mathbb{A}$ is idempotent: The operations d_a are idempotent (I) and c_a are idempotent because of the equations $(D1)$ and (I), for instance.

2. Φ preserves the composition and the identities: This is clear.

3. Φ is faithful: From Fact 1 it follows that for distinct $b, b' \in B$ the terms $d_b(x,y), d_{b'}(x,y)$ are inequivalent in $\mathbb{B}$.

4. ν_f is an interpretation: The equations $(C), (D1), (E0), (E1)$ and (I) are readily preserved. Preservation of $(D3)$ follows from the fact that f is a homomorphism: $\nu_f(c_a)(3x, 16y) = c_{f(a)}(3x, 16y) \approx_{\mathbb{B}} d_{\gamma(f(a))}(x,y) = d_{f(\alpha(a))}(x,y) = \nu_f(d_{\alpha(a)})(x,y)$. The proof for $(D7)$ is similar.

It remains to prove that Φ is full. In other words, we have to prove that every interpretation ν of $\mathbb{A}$ in $\mathbb{B}$ is of the form $\nu = \nu_f$ for some homomorphism $f : \mathcal{A} \to \mathcal{B}$. So, let $\nu : \mathrm{Term}(\Sigma_{\mathcal{A}}) \to \mathrm{Term}(\Sigma_{\mathcal{B}})$ be an interpretation.

1. Let $a \in A$. Put $t = \nu(d_a)$. The equations $(E0), (E1)$ are satisfied in $\mathbb{A}$, hence $t(t(x,y),y) \approx_{\mathbb{B}} t(x,y) \approx_{\mathbb{B}} t(x, t(x,y))$. Therefore t is of reduced height at most 1 due to Fact 2.

2. Let g_P be a permissible substitution. We have $g_P(c_a(x_0, \dots, x_{18})) \approx d_{a'}(x,y)$ in $\mathbb{A}$ for some $a' \in A$ (see the equations $(D1), (D3), (D7)$). Hence $g_P(\nu(c_a)) \approx_{\mathbb{B}} \nu(d_{a'})$. We know from the preceding step that the right hand side is a term of reduced height at most 1. From Fact 3 it follows that $\nu(c_a)$ is of reduced height at most 1.

3. The term $c_a(x_0, \dots, x_{18})$ is commutative in $\mathbb{A}$ (in the sense of (C)). Therefore the term $\nu(c_a)$ is commutative in $\mathbb{B}$. It is clear (see again Fact 1) that the only commutative terms in $\mathbb{B}$ (with height 1) are the terms $c_b(x_0, \dots, x_{18})$, $b \in B$. Thus $\nu(c_a) = c_{f(a)}(x_0, \dots, x_{18})$ for some $f(a) \in B$.

4. Since $c_a(x, 18y) \approx_{\mathbb{A}} d_a(x,y)$, we have

$$d_{f(a)}(x,y) \approx_{\mathbb{B}} c_{f(a)}(x, 18y) = \nu(c_a)(x, 18y) \approx_{\mathbb{B}} \nu(d_a).$$

Hence $\nu(d_a) = d_{f(a)}(x,y)$.

5. We have proved, that $\nu = \nu_f$. The last thing is to prove that f is a homomorphism. We have $c_a(3x, 16y) \approx_{\mathbb{A}} d_{\alpha(a)}$, hence $\nu(c_a)(3x, 16y) \approx_{\mathbb{B}} \nu(d_{\alpha(a)})$. The left hand side equals $c_{f(a)}(3x, 16y) \approx_{\mathbb{B}} d_{\gamma(f(a))}(x,y)$. The right hand side equals $d_{f(\alpha(a))}(x,y)$. Using Fact 1 we obtain $\gamma(f(a)) = f(\alpha(a))$.

6. Analogically as in the previous step, using the equation $c_a(7x, 12y) \approx_{\mathbb{A}} d_{\beta(a)}(x,y)$, we get $\delta(f(a)) = f(\beta(a))$ and the proof is complete.

6 Fact 1

Lemma 7 (Fact 1, first part) *The rewriting rules* $(D1)$, $(D3)$, $(D7)$, $(E0)$, $(E1)$, (I) *form a finitely terminating confluent rewriting system.*

PROOF. The system is finitely terminating, since each rewriting rule decreases the number of occurrences either of c_a or d_a. To prove its local confluency, it is enough to consider the critical overlaps (see Section 4). In our system, we have to consider the following cases:

1. We can apply two different rules (Di), (Dj) $(i, j \in \{1, 3, 7\})$ at the address $\emptyset$. Consider the case $(D1), (D3)$, the other possibilities are analogical. All terms $t[i]$, $i \in 19$ are equal, say, to a term t_0. We have

$$c_a(t_0, 18t_0) \xrightarrow{(D1)} d_a(t_0, t_0) \xrightarrow{(I)} t_0$$
$$c_a(3t_0, 16t_0) \xrightarrow{(D3)} d_{\alpha(a)}(t_0, t_0) \xrightarrow{(I)} t_0$$

2. We can apply the rule (Ei) $(i \in 2)$ at the address $\emptyset$ and the rule (Ej) $(j \in 2)$ at the address i. First, let $i = j = 0$. We can apply $(E0)$ at 0, thus $t[00] = d_a(t_0, t_1)$ and $t[01] = t_1$ for some terms t_0, t_1. We can apply $(E0)$ at $\emptyset$, hence $t[1] = t_1$. Therefore $t = d_a(d_a(d_a(t_0, t_1), t_1), t_1)$. But the application of both rules gives the same result:

$$d_a(d_a(d_a(t_0, t_1), t_1), t_1) \xrightarrow{(E0,1)} d_a(d_a(t_0, t_1), t_1)$$

Next, let $i = 0, j = 1$. We can apply $(E1)$ at 0, hence $t[01] = d_a(t_0, t_1)$ and $t[00] = t_0$. We can apply $(E0)$ at $\emptyset$, hence $t[1] = d_a(t_0, t_1)$. Thus $t = d_a(d_a(t_0, d_a(t_0, t_1)), d_a(t_0, t_1))$. We have

$$d_a(d_a(t_0, d_a(t_0, t_1)), d_a(t_0, t_1)) \xrightarrow{(E0)} d_a(t_0, d_a(t_0, t_1)) \xrightarrow{(E1)} d_a(t_0, t_1)$$
$$d_a(d_a(t_0, d_a(t_0, t_1)), d_a(t_0, t_1)) \xrightarrow{(E1)} d_a(d_a(t_0, t_1), d_a(t_0, t_1)) \xrightarrow{(I)} d_a(t_0, t_1)$$

The two cases $i = 1, j = 0, 1$ are symmetric.

3. We can apply (Ei), $i \in 2$ at $\emptyset$ and (I) at i. In this case $t = d_a(d_a(t_0, t_0), t_0)$ or $t = d_a(t_0, d_a(t_0, t_0))$ which can be rewritten to t_0. $\square$

Recall that the reduced form of a term t in this rewriting system is denoted by $\mathrm{Red}(t)$.

Lemma 8 (Fact 1, second part) *Let s, t be terms. Then $s \approx_{\mathbb{A}} t$ if and only if* $\mathrm{Red}(s) \sim_{\mathbb{A}} \mathrm{Red}(t)$, *where $\sim_{\mathbb{A}}$ is the equational theory based on (C).*

PROOF. Only the "only if" part is nontrivial. Let $s \approx_{\mathbb{A}} t$.

Let $\mathcal{S} = \{(Di), (Ej), (I), i \in \{1,3,7\}, j \in 2\}$ and $\equiv$ denote the equational theory generated by $\mathcal{S}$. Let p_0, p_1, p_2 be terms. Observe that if $p_0 \overset{(C)}{\longleftrightarrow}_1 p_1 \overset{\mathcal{S}}{\longleftrightarrow}_1 p_2$, then also $p_0 \overset{\mathcal{S}}{\longleftrightarrow}_1 p_3 \overset{(C)}{\longleftrightarrow}_1 p_2$ for some term p_3. Hence a derivation of $s \approx_{\mathbb{A}} t$ can be rearranged to obtain a derivation of $s \equiv s_0 \sim_{\mathbb{A}} t$, where s_0 is a term. From the previous lemma and Section 4, we know that $s \overset{\mathcal{S}}{\longrightarrow} \mathrm{Red}(s) \overset{\mathcal{S}}{\longleftarrow} s_0 \sim_{\mathbb{A}} t$. After further rearrangement we get $s \overset{\mathcal{S}}{\longrightarrow} \mathrm{Red}(s) \sim_{\mathbb{A}} s_1 \overset{\mathcal{S}}{\longleftarrow} t$ for some term s_1. Clearly, every term $\sim_{\mathbb{A}}$-equivalent to a reduced term is reduced, thus $s_1 = \mathrm{Red}(t)$. □

7 Fact 2

All terms in this section will be over $\{x, y\}$ in the signature $\Sigma_{\mathcal{A}}$.

Lemma 9 (Fact 2) *Let t be a term such that $t(x, t(x, y)) \approx_{\mathbb{A}} t(x, y)$, $t(t(x, y), y) \approx_{\mathbb{A}} t(x, y)$. Then t is of reduced height at most 1.*

PROOF. Striving for a contradiction, suppose that t is a reduced term with $\mathrm{ht}(t) > 1$ satisfying the equations. Since $\mathbb{A}$ is idempotent, t contains both variables x, y.

Let f_x denote the substitution sending x to t and y to y. Symmetrically, let f_y denote the substitution sending x to x and y to t. The equation $t(x, t(x, y)) \approx_{\mathbb{A}} t(x, y)$ means $f_y(t) \approx_{\mathbb{A}} t$. The equation $t(t(x, y), y) \approx_{\mathbb{A}} t(x, y)$ means $f_x(t) \approx_{\mathbb{A}} t$.

Claim 10 *Let s_1, s_2 be terms. If $f_x(s_1) = f_x(s_2)$, then $s_1 = s_2$. If $f_y(s_1) = f_y(s_2)$, then $s_1 = s_2$.*

PROOF. Assume $f_x(s_1) = f_x(s_2)$ (the second case is symmetric). Assume $\mathrm{ht}(s_1) \leq \mathrm{ht}(s_2)$. We proceed by induction on $\mathrm{ht}(s_1)$. First, let $s_1 = y$. Then $f_x(s_1) = y$ and clearly $s_2 = y$. Next, suppose $s_1 = x$, $s_2 \neq x$. Then $f_x(s_1) = t$. If s_2 doesn't contain x then clearly $f_x(s_1) \neq f_x(s_2)$. If s_2 contains x, then $\mathrm{ht}(f_x(s_2)) > \mathrm{ht}(t) = \mathrm{ht}(f_x(s_1))$, thus $f_x(s_1) \neq f_x(s_2)$.

The induction step is trivial. □

Claim 11 *Let s_1, s_2 be terms, $s_2 \subseteq t$, $s_2 \neq t$, $f_x(s_1) = s_2$. Then $s_1 = y$.*

PROOF. Evident. $\square$

Claim 12 *If $f_x(t)$ is not reduced, then $t = d_a(s, y)$ or $t = d_a(y, s)$, where $a \in A$ and s is a term. If $f_y(t)$ is not reduced, then $t = d_a(s, x)$ or $t = d_a(x, s)$, where $a \in A$ and s is a term.*

PROOF. We prove only the first part, the second part being symmetric.

Suppose that we can apply a rewriting rule to $f_x(t)$ at an address A. Since t is reduced, A is a valid address of t and A is not an address of a leaf of t.

We can not apply $(D1), (D3), (D7), (I)$ at A: The term t is reduced, so, if one of these rules can be applied to $f_x(t)$, we have $f_x(t[Ai]) = f_x(t[Aj]))$ for some $i, j \in 19$ such that $t[Ai] \neq t[Aj]$, which contradicts Claim 10.

Suppose, we can apply $(E0)$ at A, hence $t[A] = d_a(t_0, t_1)$. If $t_0 \neq x$, we have $t_0 = d_a(t_2, t_3)$ (because $(E0)$ can be applied to $f_x(t)$ at A), and $t_1 \neq t_3$ (because t is reduced). But $f_x(t_1) = f_x(t_3)$ (again, because we can apply $(E0)$ to $f_x(t)$ at A), which contradicts Claim 10. So $t_0 = x$. Then $t = d_a(s_0, s_1)$ and $s_1 = f_x(t_1)$. By Claim 11, $t_1 = y$, hence $s_1 = y$. Together $t = d_a(s_0, y)$.

Suppose, we can apply $(E1)$ at A, hence $t[A] = d_a(t_0, t_1)$. As in the previous paragraph $t_1 = x$, $t = d_a(s_0, s_1)$ and $s_0 = f_x(t_0)$. Hence $t_0 = y$ and $s_0 = y$. $\square$

Since $f_x(t)$, $f_y(t)$ are not reduced (because $\mathrm{ht}(f_x(t)), \mathrm{ht}(f_y(t)) > \mathrm{ht}(t)$), we have either $t = d_a(x, y)$ or $t = d_a(y, x)$ by Claim 12, a contradiction with $\mathrm{ht}(t) > 1$. $\square$

8 Fact 3

This is the most technical part of the proof. The longest part is an examination of terms of height 2 over $\{x_0, \ldots, x_{18}\}$. For those readers who don't want to read the whole proof, we would like to explain the following:

- **Why 19-ary operations, why 1,3,7?** We will need the properties of those numbers stated in Claims 14 and 15 several times, for example 17.E.4.
- **Why $\mathbf{Alg}_*(1,1)$ instead of $\mathbf{Alg}(1,1)$?** We will use the property of algebras in $\mathbf{Alg}_*(1,1)$ in the proof of 17.E.5.

In this section, all terms are in the signature $\Sigma_{\mathcal{A}}$.

Recall that

$$\mathcal{P} = \{1, 3, 7, 12, 16, 18\}.$$

Let $P \subseteq \{x_0, x_1, \ldots, x_{18}\}, |P| \in \mathcal{P}$. The substitution sending all variables in P to x and all other variables to y is called permissible substitution.

Fact 3 can be formulated as follows:

Lemma 13 (Fact 3) *Let t be a reduced term over $\{x_0, \ldots, x_{18}\}$ of height at least 2. Then there exists a permissible substitution g_P such that $g_P(t)$ is of reduced height at least 2.*

PROOF. We proceed by induction on $\mathrm{ht}(t)$ starting from $\mathrm{ht}(t) = 2$. First, we prove the induction step. Let $t_0 = t[i], i \in 19$ be of height at least $\mathrm{ht}(t) - 1 \geq 2$ (it is reduced, since t is). From the induction hypotheses, we can find a permissible substitution g_P such that $\mathrm{ht}(g_P(t_0)) \geq 2$. Put $s_0 = \mathrm{Red}(g_P(t_0))$ and let s be the term obtained by taking the term $g_P(t)$ and applying all possible rewriting rules, but not at the root. We have $\mathrm{ht}(s) \geq 3$.

If $s = d_a(s_0, s_1)$ or $s = d_a(s_1, s_0)$, the only possible rules, which can be applied, are $(E0), (E1), (I)$. In each of these cases $\mathrm{Red}(s) = s_i$, where $\mathrm{ht}(s_i) \geq \mathrm{ht}(s_{1-i})$. Hence $\mathrm{ht}(\mathrm{Red}(s)) \geq 2$.

if $s = c_a(\ldots, s_0, \ldots)$, the only possible rules are $(D1), (D3), (D7)$. After applying one of these rules, we obtain a term of the form from the last paragraph, and again $\mathrm{ht}(\mathrm{Red}(s)) \geq 2$.

It remains to prove the first step. So, we assume $\mathrm{ht}(t) = 2$ and we shall find a permissible substitution g_P such that $g_P(t)$ is of reduced height 2.

The following properties of $\mathcal{P}$ will be needed.

Claim 14 *If $i, j \in \mathcal{P}$, then $i + j \notin \mathcal{P}$.*

Claim 15 *If $i, j, k, l \in \mathcal{P}$ and $19 > i + j = k + l$, then $\{i, j\} = \{k, l\}$.*

Let

$$\begin{aligned} S &= \{t[i] \mid i \in 19 \text{ is a valid address of } t\} \\ g_P(S) &= \{\mathrm{Red}(g_P(s)) \mid s \in S\} \end{aligned}$$

Claim 16 *If g_P is a permissible substitution such that $g_P(S)$ satisfies one the following conditions $(R1-3)$, then $\mathrm{ht}(g_P(t))$ is of reduced height 2.*

($R1$) $g_P(S)$ *contains two different terms of height 1.*
($R2$) $g_P(S)$ *contains two different terms, one of which is of the form* $c_a(\ldots)$.
($R3$) $g_P(S)$ *contains three pairwise different terms.*

PROOF. Clear. □

Claim 17 *If one of the following set* H *of terms is contained in* S*, then there exists a permissible substitution such that* $g_P(S)$ *satisfies one of the conditions* $(R1-3)$. *(The numbers* e_i *in* (B) *mean the number of occurrences of* x_i*, similarly in the other rows.)*

(A) $\{d_a(x_i, x_j), d_{a'}(x_k, x_l)\}$, *if* $i \neq k$, $j \neq l$ *or* $a \neq a'$.
(B) $\{c_a(e_0x_0, e_1x_1, \ldots, e_{18}x_{18}), x_i\}$, *if there exists* $j \in 19$ *such that* $e_j \neq 1$.
(C) $\{c_a(e_0x_0, e_1x_1, \ldots, e_{18}x_{18}), d_{a'}(x_i, x_j)\}$.
(D) $\{c_a(e_0x_0, e_1x_1, \ldots, e_{18}x_{18}), c_{a'}(e_0x_0, e_1x_1, \ldots, e_{18}x_{18})\}$, *if* $a \neq a'$.
(E) $\{c_a(e_0x_0, e_1x_1, \ldots, e_{18}x_{18}), c_{a'}(f_0x_0, f_1x_1, \ldots, f_{18}x_{18})\}$, *if there exists* $i \in 19$ *such that* $e_i \neq f_i$.
(F) $\{d_a(x_i, x_j), x_k, x_l\}$, $k \neq l$.
(G) $\{c_a(e_0x_0, \ldots, e_{18}x_{18}), x_i, x_j\}$, $i \neq j$.
(H) $\{d_a(x_i, x_j), d_{a'}(x_k, x_l), x_m\}$, *if* $i \neq k$ *or* $j \neq l$.

PROOF. The proof is shown in the table.

	Assumption	$P =$	$g_P(H) =$
A.1	$a \neq a'$, $i = k$	$\{x_i\}$	$\{d_a(x, y), d_{a'}(x, y)\}$ (R1)
A.2	$a \neq a'$, $j = l$	$\{x_j\}$	$\{d_a(y, x), d_{a'}(y, x)\}$ (R1)
A.3	$i \neq k, j \neq l$	$\{x_i, x_l, x_o\}$	$\{d_a(x, y), d_{a'}(y, x)\}$ (R1)
B.1	$(\exists k)\ 0 \neq e_k \notin \mathcal{P}$	$\{x_k\}$	$\{c_a(e_kx, \ldots y), \ldots\}$ (R2)
B.2	$(\exists k, l)\ e_k, e_l \in \mathcal{P}$	$\{x_k, x_l, x_p\}$	$\{c_a((e_k + e_l)x, \ldots y), \ldots\}$ 14 (R2)
C.1	$(\exists k)\ e_k \neq 1$	Similar to B	
C.2	otherwise	$\{x_j\}$	$\{d_a(x, y), d_{a'}(y, x)\}$ (R1)
D.1	$(\exists i)\ e_i \neq 1$	Similar to B	
D.2	otherwise	$\{x_0\}$	$\{d_a(x, y), d_{a'}(x, y)\}$ (R1)
F.1	$i = k$	$\{x_i\}$	$\{d_a(x, y), x, y\}$ (R3)
F.2	$i = l$ or $j \in \{k, l\}$	Similar to F.1.	

F.3	otherwise	$\{x_i, x_k, x_o\}$	$\{d_a(x,y), x, y\}$ (R3)
G.1	$(\exists k)\ e_k \neq 1$	Similar to B	
G.2	otherwise	$\{x_i\}$	$\{d_a(x,y), x, y\}$ (R1)
H.1	$i \neq k, j \neq l$	See(A)	
H.2	$i \neq k, j = l, m = i$	$\{x_i\}$	$\{d_a(x,y), y, x\}$ (R3)
H.3	$i \neq k, j = l, m = k$	$\{x_k\}$	$\{y, d_{a'}(x,y), x\}$ (R3)
H.4	$i \neq k, j = l, m \notin \{i,k\}$	$\{x_i, x_j, x_o\}$	$\{x, d_{a'}(y,x), y\}$ (R3)
H.5	$j \neq l, i = k$	Similar to H.2-4	

For example row A.1 reads as follows: If $a \neq a'$, let $P = \ldots$. We have $g_P(H) = \ldots$ and this satisfies the condition (R1) from the last claim. The letters i, j, k, l, m denote elements of 19. In rows A.3,F.3,H.4, o is an arbitrary element of 19 distinct from i, j, k, l. In B.2, $p \in 19$ is such that $e_p = 0$. In B.2 we need Claim 14 to know that the term $c_a((e_k + e_l)x, \ldots y)$ is reduced. Note also that, for example, $i \neq j$, $k \neq l$ in the case (A), because H is a set of reduced terms; $e_k + e_l < 19$ in B.2 for the same reason, etc.

It remains to prove (E). Let us continue writing the table.

E.1	$(\exists j)\ e_j \neq f_j, 0 \neq e_j \notin \mathcal{P}$	$\{x_j\}$	$\{c_a(e_j x, \ldots y), \ldots\}$ (R2)
E.2	$(\exists j,k,l)\ e_j = e_k = e_l = 1$ $f_j = 0, f_k, f_l \in \mathcal{P}$	$\{x_j, x_k, x_l\}$	$\{d_{\ldots}(x,y),$ $c_{a'}((f_k + f_l)x, \ldots)\}$ 14 (R2)
E.3	$(\exists j,k,l)\ e_j = e_k = f_l = 0$ $f_j, f_k, e_l \in \mathcal{P}$	$\{x_j, x_k, x_l\}$	$\{d_{\ldots}(\ldots),$ $c_{a'}((f_j + f_k)x, \ldots)\}$ 14 (R2)

First suppose that for all j either $e_j \neq 0$ or $f_j \neq 0$. If $e_j = 1$ for all j (or $f_j = 1$ for all j), then we can use either E.1 or E.2 (eventually with e and f interchanged). Otherwise we can use E.1 or E.3 (in case there is more than one zero among the numbers $e_0, \ldots$ or $f_0, \ldots$) or E.1 (in case that $(\exists j)\ 2 = e_j \neq f_j$ or $2 = f_j \neq e_j$) or E.2.

Now, assume $e_j = f_j = 0$ for some j and $e_k = f_k \neq 0$ for some k and take i such that $e_i \neq f_i$.

E.4	$a \neq a'$, $e_k \notin \mathcal{P}$	$\{x_k\}$	$\{c_a(e_k x, \ldots y), c_{a'}(e_k x, \ldots)\}$	
E.5	$a = a'$, $e_i, f_i \in \mathcal{P}$	$\{x_i\}$	$e_i < 10, f_i > 10 \ldots \{d(x,y), d(y,x)\}$	
			$e_i > 10, f_i < 10 \ldots \{d(y,x), d(x,y)\}$	
			$e_i, f_i < 10 \ldots \{d_r(x,y), d_s(x,y)\}$	
			$r \neq s$ from properties of $\mathbf{Alg}_*(1,1)$	
			$e_i, f_i > 10 \ldots \{d_r(y,x), d_s(y,x)\}$	
			$r \neq s$ from properties of $\mathbf{Alg}_*(1,1)$	
E.6	$e_i, f_i, e_k \in \mathcal{P}$	$\{x_i, x_j, x_k\}$	$\{c_a((e_i + e_k)x, \ldots),$	14
			$c_{a'}((f_i + f_k), \ldots)\}$	
E.7	$e_i = 0, f_i \in \mathcal{P}$	$\{x_i, x_j, x_k\}$	$e_k \in \mathcal{P} \ldots \{\ldots, c_{a'}((f_i + f_k)x, \ldots)\}$	
			$e_k \notin \mathcal{P} \ldots \{c_a(e_k x, \ldots), \ldots\}$	14 ↑

We can further assume $e_i, f_i \in \mathcal{P} \cup \{0\}$ (otherwise use E.1), $e_i \neq 0$ (otherwise E.7), $f_i \neq 0$ (otherwise analogue of E.7), $a \neq a'$ (otherwise E.5), $e_k \in \mathcal{P}$ (otherwise E.4). Now, we can use E.6.

The last case is that $e_j = f_j = 0$ and $e_k \neq f_k$ for all k for which $e_k \neq 0$ or $f_k \neq 0$. We can assume $e_k, f_k \in \mathcal{P} \cup \{0\}$ for all k (otherwise E.1). We can find pairwise distinct l, m, n such that $e_l, e_m, e_n \in \mathcal{P}$ (otherwise the term $c_a(e_0 x_0, \ldots)$ is not reduced). It is easy to see that either $\{e_l, e_m\} \neq \{f_l, f_m\}$ or $\{e_l, e_n\} \neq \{f_l, f_n\}$ or $\{e_m, e_n\} \neq \{f_m, f_n\}$. In the first case put $P = \{x_j, x_l, x_m\}$. Then $g_P(H)$ will satisfy (R2) according to Claim 15 (or Claim 14, if $0 \in \{f_l, f_m\}$). The other two cases are analogical. □

Now, we are ready to finish the proof of Fact 3.

The first possibility is $t = d_a(t_0, t_1)$. The only remaining cases, where we can't use Claim 17 are in the following table (o is again an element of 19 distinct from i, j, k; in the last row, we can assume $a \neq a'$, otherwise t is not reduced; the case $t_0 = x_k$, $t_1 = d_{a'}(x_i, x_j)$ can be solved analogically as in the last two rows).

Case	$P =$	$\mathrm{Red}(g_P(t)) =$
$t_0 = c_{a'}(x_0, \ldots, x_{18})$	$\{x_i\}$	$d_a(d_{a'}(x,y), x)$

$t_1 = x_i$		
$t_0 = x_i$	$\{x_o\}$	$d_a(y, d_{a'}(x, y))$
$t_1 = c_{a'}(x_0, \ldots, x_{18})$		
$t_0 = d_{a'}(x_i, x_j)$	$\{x_i, x_k, x_o\}$	$d_a(d_{a'}(x, y), x)$
$t_1 = d_{a'}(x_i, x_k), j \neq k$		
$t_0 = d_{a'}(x_i, x_j)$	$\{x_i, x_j, x_o\}$	$d_a(x, d_{a'}(y, x))$
$t_1 = d_{a'}(x_k, x_j), i \neq k$		
$t_0 = d_{a'}(x_i, x_j)$	$\{x_j\}$	$d_a(d_{a'}(y, x), y)$
$t_1 = x_k, k \neq j$		
$t_0 = d_{a'}(x_i, x_j)$	$\{x_i\}$	$d_a(d_{a'}(x, y), y)$
$t_1 = x_j, a \neq a'$		

The second possibility is $t = c_a(t_0, \ldots, t_{18})$.

Suppose that there exists $i \in 19$ such that $t_i = c_{a'}(\ldots)$. The only case, where we can't apply Claim 17 is $t = c_a(jc_{a'}(x_0, x_1, \ldots, x_{18}), \ldots x_k)$ for some $j \notin \mathcal{P}$. Let $P = \{x_0\}$. We have $g_P(t) = c_a(jd_{a'}(x, y), \ldots)$.

The remaining possibilities are (up to a permutation of variables)

$$t = c(e_0 x_0, e_1 d(x_0, x_1), \ldots, e_{18} d(x_0, x_{18}))$$

and

$$t = c(e_0 x_0, e_1 d(x_1, x_0), \ldots, e_{18} d(x_{18}, x_0)),$$

where the indices of c and d are arbitrary, $e_i \in 19$. Consider the first case, the second one is similar.

$(\exists i)\ 0 \neq e_i \notin \mathcal{P}$	$\{x_i\}$	
$i = 0$		$c(e_0 x, \ldots d(x, y))$
$i \neq 0$		$c(\ldots y, e_i d(y, x))$
$(\exists i, j, k)\ e_i = 0, e_j, e_k \in \mathcal{P}$	$\{x_i, x_j, x_k\}$	
$0 \in \{i, j, k\}$		$c((e_j + e_k)x, \ldots d(x, y))$ (14)
$0 \notin \{i, j, k\}$		$c(\ldots y, (e_j + e_k)d(y, x))$
$(\forall i)\ e_i = 1$	$\{x_1, x_2, x_3\}$	$d(d(y, x), y)$

This finishes the proof of Fact 3. □

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