Open Problems from the Workshop on Algebra and CSPs

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This is a list of open problems from the algebra workshop which was a part of the Summer Thematic Program on the Mathematics of Constraint Satisfaction held in the Fields Institute from July to August 2011.

The thematic program website is

http://www.fields.utoronto.ca/programs/scientific/11-12/constraint/index.html

The workshop website is

http://www.fields. . . . /constraint/algebra/index.html

It contains links to some papers and write-ups.

Finally, at

http://www.fields.utoronto.ca/audio/# algebra

you can find slides and audio for the lectures.

The list was extracted from the lectures and from the open problem session. The parenthesis contains the name of the lecturer or the person(s) who formulated the question.

1 CSPs over finite templates

In this section $\mathbb{A}$ denotes a finite relational structure and $\mathbf{A}$ denotes a finite algebra.

1.1 Dichotomy

An algebra is globally tractable, if CSP(Inv($\mathbf{A}$)) is tractable. An algebra is locally tractable, if CSP($\mathbb{A}$) is tractable for every finite subset $\mathbb{A}$ of Inv($\mathbf{A}$).

Problem 1 (Algebraic Dichotomy Conjecture; Bulatov, Jeavons, Krokhin)**.** *Let $\mathbf{A}$ be a finite algebra such that $V(\mathbf{A})$ omits 1 (i.e. $\mathbf{A}$ is Taylor). Is* CSP($\mathbf{A}$) *locally/globally tractable?*

The following is an important special case:

Problem 2. *Let $\mathbf{A}$ be a finite algebra such that $V(\mathbf{A})$ is congruence modular. Is $\mathbf{A}$ locally/globally tractable? (Interesting special case: $V(\mathbf{A})$ is congruence 3-permutable.)*

Positive answer to Valeriote's conjecture (Problem 17) would provide an affirmative answer to the local version of this problem. The global case seems to require a different algorithm for few subpowers. In particular:

Problem 3. *Find a different algorithm for* CSP($\mathbf{A}$) *where $\mathbf{A}$ is of the type "Maltsev over Maltsev", i.e. $\mathbf{A}$ has a congruence α such that $\mathbf{A}/\alpha$ is Maltsev and all α-blocks are Maltsev.*

Maróti (see his talk) provided an algorithm in the case that $\mathbf{A}$ is "Maltsev over bounded width". Can the previous problem be solved by some modification of his algorithm? The following case is also of interest:

Problem 4 (Maróti)**.** *Can Maróti's algorithm be modified for algebras of the type "bounded width over bounded width"?*

In the following problem, by a *template* we mean a set of idempotent algebras closed under taking subalgebras and idempotent images. We say that an algebra $\mathbf{B}$ *can be eliminated* if CSP($\mathcal{B}$) is tractable for all templates $\mathcal{B}$ for which $\mathcal{B} \setminus \mathbf{B}$ is also a template and CSP($\mathcal{B} \setminus \mathbf{B}$) is tractable.

Maróti proved the following theorem (Elimination Theorem)

Theorem 5. *Let $\mathbf{B}$ be an algebra and $t(x, y)$ be a binary term such that the unary maps $y \to t(b, y)$, $b \in B$, are idempotent and not surjective. Let C be the set of elements $c \in B$ for which $x \to t(x, c)$ is a permutation. If C generates a proper subuniverse of $\mathbf{B}$, then $\mathbf{B}$ can be eliminated.*

Problem 6 (Maróti)**.** *Can we avoid the condition* $\mathrm{Sg}(C) \neq B$ *in the elimination theorem?*

1.2 Finer complexity classification

Problem 7 (Larose and Tesson). *Let $\mathbb{A}$ be a relational structure such that $V(\mathrm{Pol}(\mathbb{A}))$ is congruence join semi-distributive (omits $1, 2, 5$). Is $\neg\,\mathrm{CSP}(\mathbb{A})$ definable in linear datalog?*

It is known to be true if $\mathbb{A}$ has a majority polymorphism (Dalmau, Krokhin), and, more generally, if $\mathbb{A}$ has a near unanimity polymorphism (Barto, Kozik, Willard, see Kozik's talk).

Problem 8 (Larose and Tesson). *Let $\mathbb{A}$ be a relational structure such that $V(\mathrm{Pol}(\mathbb{A}))$ is n-permutable and $\mathrm{SD}(\vee)$ (omits $1, 2, 4, 5$). Is $\neg\,\mathrm{CSP}(\mathbb{A})$ definable in symmetric datalog?*

Known for $n = 2$ (Dalmau, Larose).

Problem 9 (Dalmau). *Is it true that $\mathrm{CSP}(\mathbb{A}) \in NL$ implies $\neg\,\mathrm{CSP}(\mathbb{A})$ in linear datalog? (modulo some natural complexity-theoretic assumption)*

Problem 10 (Dalmau). *Is it true that $\mathrm{CSP}(\mathbb{A}) \in L$ implies $\neg\,\mathrm{CSP}(\mathbb{A})$ in Symmetric Datalog? (modulo some natural complexity-theoretic assumption)*

Problem 11 (Krokhin). *Assume $\mathbb{A}$ is a core such that $V(\mathrm{Pol}(\mathbb{A}))$ is n-permutable. Can $\mathrm{CSP}(\mathbb{A})$ be put in a (potentially) smaller complexity class than PTIME (e.g. Mod_pL for some p)?*

This is known to be true for Boolean case. The case $n = 2$ (e.g. Maltsev) is a good starting point.

1.3 Compact representations

Problem 12 (Maróti). *Is there a polynomial time algorithm to compute a compact representation of $\mathrm{Sg}(R \cup S)$ from compact representations of $R, S \leq \mathbf{A}^n$, where $\mathbf{A}$ is an algebra with few subpowers?*

Problem 13 (Dyer). *What is the exact complexity of computing compact representation of the set of solutions of an instance over a relational structure with Maltsev polymorphism?*

We know it is no worse than $O(n^5)$, where n is the number of variables in the input (not necessarily distinct). This computation also dominates the counting algorithm of Dyer and Richerby (see Dyer's talk).

1.4 Other complexity questions

Problem 14 (Bodirsky, Dalmau, Martin, Pinsker). *Solve Problem 1 in the case that unary term operations of* $\mathbf{A}$ *form a transitive permutation group.*

Problem 15 (Hell). *We say that* $\mathbb{A}$ *is hereditary tractable if* $\mathbb{A}$ *is tractable and all its substructures are tractable as well. When is* $\mathbb{A}$ *hereditary tractable?*

Problem 16 (Chen). *Find an algebraic characterization of applicability of SAC (singleton path consistency) and PAC (peek arc consistency)?*

An instance is SAC if it is 1-minimal and when we fix the value of a variable to an admissible element of the domain, the 1-minimality algorithm will still succeed (i.e. will not return empty constraint relations). An instance is PAC if it is 1-minimal and for any variable there exists an element of the domain such that, when we fix the value of the variable to the chosen element, the 1-minimality algorithm will succeed.

2 Universal algebra

2.1 Collapses of Maltsev conditions

A general type of questions is of the form whether one Maltsev condition implies another one when one restricts to

1. General clones (varieties)
2. Clones of ω-categorical structures
3. Locally closed clones
4. Locally finite clones
5. Clones on finite sets
6. Finitely related clones (polymorphisms of finite relational structures of finite signature, or, without loss of generality, polymorphisms of relational structures with at most binary relations)
7. Polymorphisms of digraphs (this is not too far from previous item by a reduction by Delić, Jackson and Niven)

8. Polymorphisms of special types of digraphs (like reflexive digraphs, smooth digraphs, undirected graphs, oriented trees, etc.)

9. Conservative version of the previous items. It is known that for conservative binary relational structure $\mathbb{A}$, $\mathrm{Pol}(\mathbb{A})$ is Taylor implies that $\mathrm{Pol}(\mathbb{A})$ generates a meet semi-distributive variety (Hell, Rafiey and Kazda, see his talk).

The most important problem is the following:

Problem 17 (Valeriote's conjecture, Edinburgh conjecture)**.** *Does every finitely related algebra in a congruence modular variety have few subpowers? (Interesting special case:* 3*-permutable varieties.)*

For a recent progress on this conjecture see McKenzie's talk. A positive answer would imply trichotomy for PPEQ and PPCON (see Chen's talk).

Maróti and Zádori proved that for every reflexive digraph $\mathbb{A}$, if $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$ generates a congruence modular variety then $\mathbf{A}$ has a near unanimity operation and also $\mathbf{A}$ has totally symmetric idempotent operations of all arities (see Zádori's talk). Can this be generalized to smooth digraphs? Or, at least, to smooth digraphs of algebraic length one?

Problem 18. *Let* $\mathbb{A}$ *be a smooth digraph (of algebraic length one) such that the variety generated by* $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$ *is congruence modular. Does* $\mathbf{A}$ *always have a near unanimity operation? Does* $\mathbf{A}$ *always have totally symmetric idempotent operations of all arities?*

The next two questions are motivated by the research on special triads and polyads (Barto, Bulín, Kozik, Maróti, Niven), where the answer is positive.

Problem 19. *Let* $\mathbb{A}$ *be an oriented tree such that* $\mathrm{Pol}(\mathbb{A})$ *is a Taylor algebra. Does* $\mathrm{Pol}(\mathbb{A})$ *generate a meet semi-distributive variety?*

Problem 20. *Let* $\mathbb{A}$ *be an oriented tree such that* $\mathrm{Pol}(\mathbb{A})$ *has a binary commuatative idempotent operation (wnu of arity* 2*). Does* $\mathrm{Pol}(\mathbb{A})$ *have totally symmetric idempotent operations of all arities?*

2.2 Deciding Maltsev conditions

Problem 21. *What is the computational complexity of deciding whether*

1. $V(\mathbf{A})$ *is congruence permutable (=* $\mathbf{A}$ *has a Maltsev operation),*
2. $V(\mathbf{A})$ *is congruence singular,*
3. $V(\mathbf{A})$ *is congruence uniform,*
4. $\mathbf{A}$ *has few subpowers,*
5. $\mathbf{A}$ *has totally symmetric operations (TSI) of all arities.*

There are two versions of the problem. The input can either be an algebra $\mathbf{A}$*, or a relational structure* $\mathbb{A}$ *(and the question is asked for the algebra* $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$*).*

How does the complexity change if we assume that $\mathbf{A}$ *is idempotent?*

Many of similar questions are answered in a paper by Valeriote and Freese.

A polynomial time algorithm for deciding whether a finite idempotent algebra has few subpowers was given in McKenzie's talk. Dyer and Richerby (see Dyer's talk) have shown that deciding congruence uniformity for relational structures is reducible to the graph isomorphism problem. Relational nonidempotent version of the TSI problem is known to be NP-hard (Larose, Loten, Tardiff).

2.3 Absorption

Problem 22 (Barto)**.** *Is the following problem decidable? Input is a finite algebra* $\mathbf{A}$ *and a subset* B*. Question is whether* $B \lhd \mathbf{A}$*.*

During the summer program Horowitz and Valeriote has shown that the answer is positive for $|B| = 1$. This generalizes Maróti's result that near unanimity is decidable. Both results provide very large upper bounds on the arity of the operation providing the absorption.

Problem 23. *Find a (better) upper bound on the arity of the operations providing an absorption in an algebra* $\mathbf{A}$ *(in particular, near unanimity operation).*

Problem 24 (Barto). *Is the following problem decidable? Input is a relational structure $\mathbb{A}$ and a subset B. Question is whether $B \lhd \mathrm{Pol}(\mathbb{A})$.*

During the summer program Bulín has shown (see his talk) that if $\mathrm{Pol}(\mathbb{A})$ is in a meet semi-distributive variety, then every absorption is witnessed by a term of arity bounded by a certain number depending on $|A|$ and the maximal arity k of relation in $\mathbb{A}$ (doubly exponential in $|A|^k$).

Problem 25. *Find a (better) upper bound on the arity of the operations providing an absorption in* $\mathrm{Pol}(\mathbb{A})$ *for a relational structure* $\mathbb{A}$ *(in particular, near unanimity operation).*

The two simplified proofs of the dichotomy for conservative CSPs (see Bulatov's and Barto's talks) are quite similar, but use different notions. Are they somehow related?

Problem 26 (Bulatov). *Is there a connection between as-components and minimal absorbing subuniverses?*

2.4 Other UA question

Problem 27 (Barto). *Characterize finite algebras (or their clones) in the variety generated by all conservative Taylor algebras of a given type.*

Problem 28 (Maróti). *Find obstructions for congruence modular structures.*

See Larose's talk for the definition of obstructions.

3 CSPs over infinite templates

In this section $\mathbb{A}$ is a countably infinite relational structure with finitely many relations.

The main goal is to solve the following problem:

Problem 29 (Bodirsky). *Prove the dichotomy for CSPs over reducts of finitely bounded homogeneous structures assuming the dichotomy for finite CSPs.*

The following problems were proposed in the talks of Bodirsky and Pinsker to attack the problem.

Problem 30 (Bodirsky). *Show that the complexity of* $\mathrm{CSP}(\mathbb{A})$ *only depends on the variety generated by* $\mathrm{Pol}(\mathbb{A})$*, where* $\mathbb{A}$ *is finitely bounded homogeneous relational structure. If not, is the complexity determined by the variety plus the natural topology?*

Problem 31 (Bodirsky). *Do all finitely bounded homogeneous structures have the Finite Dimension Property?*

Problem 32 (Pinsker). *Is every structure which is homogeneous and finitely bounded a reduct of a structure which is ordered Ramsey, homogeneous, and finitely bounded?*

Problem 33 (Bodirsky). *Let* $\mathbb{A}$ *be a reduct of a finitely bounded homogeneous structure. Is* $\mathrm{CSP}(\mathbb{A})$ *necessarily in P when one of the following conditions holds?*

- *For all* n *there is a canonical* $f \in \mathrm{Pol}(\mathbb{A})$ *such that for all* $\pi \in S_n$ *there is* $\alpha \in \mathrm{Aut}(\mathbb{A})$ *satisfying*

$$f(x_1, \ldots, x_n) = \alpha f(x_{\pi(1)} \ldots, x_{\pi(n)}).$$

- *There exists a ternary canonical* $f \in \mathrm{Pol}(\mathbb{A})$ *and* $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Aut}(\mathbb{A})$ *such that*

$$f(x, x, y) = \alpha_1 f(x, y, x) = \alpha_2 f(y, x, x) = \alpha_3 x.$$

- *There exists a ternary canonical* $f \in \mathrm{Pol}(\mathbb{A})$ *and* $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Aut}(\mathbb{A})$ *such that*

$$f(x, x, y) = \alpha_1 f(x, y, x) = \alpha_2 f(y, x, x) = \alpha_3 y.$$

Bodirsky and Pinsker developed a rather general method to obtain classification results over nice templates and they proved a dichotomy for reducts of the random graph (see Pinsker's talk). Some other natural cases:

Problem 34 (Pinsker). *Classify the complexity of CSP over reducts of the random partial order, the random tournament, the random* K_n*-free graph, the atomless Boolean algebra, the random lattice.*

Problem 35 (Chen). *Classify the complexity of PPEQ (equivalence problem for primitive positive formulas, see Chen's talk) and PPCON (containement problem for pp-formulas) over infinite structures.*

4 Other variants of CSPs

4.1 Counting CSPs

Problem 36 (Dyer)**.** *Can the counting algorithm be made more efficient in some natural special cases?*

Most known special cases have $O(n)$ counting algorithm, where n is the number of variables.

Problem 37 (Dyer)**.** *What can be said about counting CSPs if restrictions are placed on the instance? For example, if any variable can occur only a bounded number of times in the constraints?*

The dichotomy for counting CSPs (Bulatov; Dyer and Richerby) extends to rational nonnegative weights (Bulatov, Dyer, Goldberg, Jalsenius, Jerrum, Richerby) and to algebraic nonnegative weights (Cai, Chen, Lu).

Problem 38 (Dyer)**.** *Classify the complexity for counting CSPs with negative or complex weights.*

Problem 39 (Dyer)**.** *What is the complexity of approximate counting?*

It seems unlikely that a simple dichotomy exists, but Goldberg and Jerrum have given a trichotomy for the 2-element case. For recent progress see Bulatov's talk in the approximation workshop.

4.2 Valued CSP

Problem 40. *Classify the computational complexity of valued CSPs.*

There are two versions of this problem - we can allow infinite weights or not. The first version is at least as hard as the CSP dichotomy. One of the special cases is the following.

Problem 41 (Kolmogorov)**.** *What pairs of binary multimorphisms guarantee tractability of valued CSPs?*

Examples include join and meet on a distributive lattice, and on some non-distributive lattices (Krokhin, Larose; Kuivinen), bisubmodular functions, some tree-submodular functions (Kolmogorov).

It seems that idempotent commutative multimorphisms are especially important (see Thapper's talk).

Problem 42 (Thapper). *Find a general class $\mathcal{C}$ of binary idempotent commutative multimorphisms such that a k-ary cost function h has the multimorphism $(f,g) \in \mathcal{C}$ iff every binary function obtained from h by replacing any given $k-2$ arguments by constants has the multimorphism (f,g).*

Jeavons in his talk described a Galois correspondence for valued CSPs, the algebraic objects are so-called weighted clones.

Problem 43 (Jeavons). *Study the Boolean weighted clone lattice. As a first step, find its cardinality.*

Creed and Živný have found all the minimal weighted clones (there is 9 of them).

Problem 44 (Jeavons). *Study the weighted clone lattice for larger domains (for instance minimal elements).*

Problem 45. *Find a useful notion of a core for valued CSPs.*

4.3 Robust approximation of CSPs

Problem 46 (Dalmau). *For which relational structures $\mathbb{A}$ does there exist a robust approximation algorithm for* MixedCSP($\mathbb{A}$) *(the hard-soft constraints version of* MaxCSP($\mathbb{A}$)*)?*

Problem 47 (Dalmau, Krokhin). *Can we add equality to the template without changing robust approximability? Can we go to powers?*

Problem 48. *How to approximate satisfiable NP-hard CSPs (i.e., study $(\alpha, 1)$ approximation)?*

Problem 49. *Prove that*

$$\mathrm{CSP}(\mathbb{Q}, R_a = \{(x,y) : x - a = y\},\ a \in \mathbb{Q}\ )$$

is not robustly approximable without assuming the Unique Games Conjecture.

4.4 Other problems

Let B_n be the set of binary relations on the set $\{1, ..., n\}$ considered as a monoid with identity element $\{(1,1), ..., (n,n)\}$ and the natural relational composition. Submonoids of B_n may require exponentially many generators, as a function of n (indeed, B_n itself has this property (Devadze)). Now restrict attention to relations $E \in B_n$ satisfying the property

$$(*) \quad \forall x \ \exists y \ E(x,y) \ \wedge \ \forall y \ \exists x \ E(x,y).$$

Let the monoid of such relations be denoted X_n. We believe submonoids of X_n may require exponentially many generators, as a function of n. Finally, consider monoids M whose elements come from X_n but which additionally enjoy the property of down-closure, i.e. if $E \in M$ and F satisfies $\forall x, y \ F(x,y) \rightarrow E(x,y)$, then $F \in M$ also (F must still satisfy property (*)).

Problem 50 (Martin). *Imbued with down-closure, as well as composition, do submonoids of X_n still require exponentially many generators, or is some polynomial set sufficient (recall a linear number is sufficient for subgroups of the symmetric group S_n)?*

Let $\mathbb{A}$ be an ω-categorical structure, sPol($\mathbb{A}$) be its set of surjective polymorphisms and pH be the logic involving only both quantifiers, conjunction and equality. (The logic pH is called "positive Horn" in Model Theory, but has gone by various names in Computer Science, such as "few" and "conjunctive positive".)

Problem 51 (Bodirsky, Chen and Martin). *Is it the case that* $\mathrm{Inv}(\mathrm{sPol}(\mathbb{A})) = \langle \mathbb{A} \rangle_{\mathrm{pH}}$*, i.e. are the relations of $\mathbb{A}$ that are preserved by the surjective polymorphisms of $\mathbb{A}$ precisely the relations that are pH-definable in $\mathbb{A}$?*

Similar relationships are known to hold for most other fragments of first-order logic (existential positive with endomorphisms, Σ_1 with embeddings, first-order with automorphisms, etc.) i.e. pH appears particularly challenging. A weaker version of this connection has been given by Müller (unpublished) involving periodic polymorphisms of infinite arity. The problem has a positive answer in the case of finite structures.