# CSP lecture 17/18 winter semester – Problem Set 2

We say that CSP($\mathbb{A}$) is *polynomially reducible* to CSP($\mathbb{B}$) if there exists a polynomial–time algorithm which transforms an input $I$ of CSP($\mathbb{A}$) to an input $r(I)$ of CSP($\mathbb{B}$) so that $I$ is satisfiable iff $r(I)$ is satisfiable. In such a case, we write CSP($\mathbb{A}$) $\leq_P$ CSP($\mathbb{B}$). When CSP($\mathbb{A}$) $\leq_P$ CSP($\mathbb{B}$) $\leq_P$ CSP($\mathbb{A}$), we write CSP($\mathbb{A}$) $\sim_P$ CSP($\mathbb{B}$) and say that the two problems are *polynomially equivalent.*

Observe that if CSP($\mathbb{A}$) $\leq_P$ CSP($\mathbb{B}$) and CSP($\mathbb{B}$) in in P (ie. solvable in polynomial time), then CSP($\mathbb{A}$) in in P. Similarly, if CSP($\mathbb{A}$) $\leq_P$ CSP($\mathbb{B}$) and CSP($\mathbb{A}$) is NP–complete, then CSP($\mathbb{B}$) is NP–complete.

**Problem 1**. Let $\mathbb{A} = (\{0,1,2\}; C_0, C_1, Q)$, where

$$C_0 = \{0\}, C_1 = \{1\}, Q = \{000, 110, 120, 210, 101, 102, 201, 202, 011, 012, 021\}$$

($Q$ is a ternary relation, we omit the commas and parentheses, eg. 110 stands for (1,1,0).)

Moreover, let $\mathbb{B}$ be the relational structure $\mathbb{B} = (\{0,1\}; C_0, C_1, G_1)$ (where the notation is from the 1st problem set). Prove that CSP($\mathbb{A}$) $\sim_P$ CSP($\mathbb{B}$). (Hint: use homomorphisms $\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$).

**Problem 2**. Prove that for each finite relational structure $\mathbb{A}$ there exists a relational structure $\mathbb{B}$ such that

- there exists a homomorphism $\mathbb{A} \to \mathbb{B}$ and a homomorphism $\mathbb{B} \to \mathbb{A}$, and
- $\mathbb{B}$ is a *core*, that is, each endomorphism of $\mathbb{B}$ is an automorphism.

Deduce that we can WLOG concentrate on CSPs over cores.

Also prove that such a structure is unique up to isomorphism.

**Problem 3**. Let $\mathbb{A} = (A; R_1, R_2, R_4)$ be a relational structure, where $R_i$ is an $i$-ary relation. Let $E$ be the equality relation (ie. $E = \{(a,a) : a \in A\}$, let $S$ be the ternary relation on $A$ defined by

$$S(x,y,z) \quad \text{iff} \quad R_1(x) \wedge R_2(x,z) \wedge R_4(y,z,y,x)$$

and let $T$ be the binary relation defined by

$$T(x,y) \quad \text{iff} \quad (\exists z \in A)\ S(x,y,z)$$

Prove that

- CSP($A; R_1, R_2, R_4, E$) $\leq_P$ CSP($\mathbb{A}$)
- CSP($A; R_1, R_2, R_4, E, S$) $\leq_P$ CSP($\mathbb{A}$)
- CSP($A; R_1, R_2, R_4, E, S, T$) $\leq_P$ CSP($\mathbb{A}$)

Try to formulate a general theorem covering these particular cases.

**Problem 4**. Prove that CSP($\mathbb{A}$), CSP($\mathbb{B}$) and CSP($\mathbb{C}$) are polynomially equivalent, where

$$\mathbb{A} = (\{0,1,2\}, C_0, C_1, C_2, N), \quad N = \{0,1,2\}^2 \setminus \{(0,0),(1,1),(2,2)\}$$
$$\mathbb{B} = (\{0,1\}, S_{000}, S_{001}, S_{011}, S_{111}), \quad S_{ijk} = \{0,1\}^3 \setminus \{(i,j,k)\}$$
$$\mathbb{C} = (\{0,1\}, C_0, C_1, R), \quad R = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$$

**Problem 5**. prove that CSP($\mathbb{A}$) $\sim_P$ CSP($\{0,1,2\}, N$), where $\mathbb{A}, N$ are from the previous problem.

**Problem 6**. For each finite relational structure $\mathbb{A}$ find an input of CSP($\mathbb{A}$) whose solutions precisely correspond to endomorphisms of $\mathbb{A}$.

**Problem 7**. Let $\mathbb{A}$ be a finite *core* and let $\mathbb{B}$ be the relational structure formed from $\mathbb{A}$ by adding all the unary relations $C_a = \{a\}$, $a \in A$. Prove that CSP($\mathbb{A}$) $\sim_P$ CSP($\mathbb{B}$).

**Problem 8**. Let $\mathbb{A}$ be a relational structure such that CSP($\mathbb{A}$) is in P. Prove that there is a polynomial–time algorithm for finding a solution of CSP($\mathbb{A}$).