

CSP lecture 21/22 winter semester – Problem Set 2

Recall that $\text{CSP}(\mathbb{A})$ is *polynomially reducible* to $\text{CSP}(\mathbb{B})$ if there exists a polynomial-time algorithm that transforms an input I of $\text{CSP}(\mathbb{A})$ to an input $r(I)$ of $\text{CSP}(\mathbb{B})$ so that I is satisfiable iff $r(I)$ is satisfiable. In such a case, we write $\text{CSP}(\mathbb{A}) \leq_P \text{CSP}(\mathbb{B})$. When $\text{CSP}(\mathbb{A}) \leq_P \text{CSP}(\mathbb{B}) \leq_P \text{CSP}(\mathbb{A})$, we write $\text{CSP}(\mathbb{A}) \sim_P \text{CSP}(\mathbb{B})$ and say that the two problems are *polynomially equivalent*.

Observe that if $\text{CSP}(\mathbb{A}) \leq_P \text{CSP}(\mathbb{B})$ and $\text{CSP}(\mathbb{B})$ is in P (i.e., solvable in polynomial time), then $\text{CSP}(\mathbb{A})$ is in P. Similarly, if $\text{CSP}(\mathbb{A}) \leq_P \text{CSP}(\mathbb{B})$ and $\text{CSP}(\mathbb{A})$ is NP-complete, then $\text{CSP}(\mathbb{B})$ is NP-complete.

Problem 1. Let $\mathbb{A} = (\{0, 1, 2\}; C_0, C_1, Q)$, where

$$C_0 = \{0\}, C_1 = \{1\}, Q = \{000, 110, 120, 210, 101, 102, 201, 202, 011, 012, 021\}$$

(Q is a ternary relation, we omit the commas and parentheses, eg. 110 stands for $(1,1,0)$.)

Moreover, let \mathbb{B} be the relational structure $\mathbb{B} = (\{0, 1\}; C_0, C_1, G_1)$ (where the notation is from the 1st problem set). Prove that $\text{CSP}(\mathbb{A}) \sim_P \text{CSP}(\mathbb{B})$. (Hint: use homomorphisms $\mathbb{A} \rightarrow \mathbb{B}$ and $\mathbb{B} \rightarrow \mathbb{A}$.)

Problem 2. Prove that for each finite relational structure \mathbb{A} there exists a relational structure \mathbb{B} such that

- there exists a homomorphism $\mathbb{A} \rightarrow \mathbb{B}$ and a homomorphism $\mathbb{B} \rightarrow \mathbb{A}$, and
- \mathbb{B} is a *core*, that is, each endomorphism of \mathbb{B} is an automorphism.

Deduce that we can WLOG concentrate on CSPs over cores.

Also prove that such a structure is unique up to isomorphism.

Problem 2'. Find a relational structure \mathbb{A} such that every structure \mathbb{B} with homomorphisms $\mathbb{A} \rightarrow \mathbb{B}$ and $\mathbb{B} \rightarrow \mathbb{A}$ is *not* a core. (Hint: \mathbb{A} can be taken to be a directed graph.)

Problem 3. Let $\mathbb{A} = (A; R_1, R_2, R_4)$ be a relational structure, where R_i is an i -ary relation. Let E be the equality relation (ie. $E = \{(a, a) : a \in A\}$), let S be the ternary relation on A defined by

$$S(x, y, z) \quad \text{iff} \quad R_1(x) \wedge R_2(x, z) \wedge R_4(y, z, y, x)$$

and let T be the binary relation defined by

$$T(x, y) \quad \text{iff} \quad (\exists z \in A) S(x, y, z)$$

Prove that

- $\text{CSP}(A; R_1, R_2, R_4, E) \leq_P \text{CSP}(\mathbb{A})$
- $\text{CSP}(A; R_1, R_2, R_4, E, S) \leq_P \text{CSP}(\mathbb{A})$
- $\text{CSP}(A; R_1, R_2, R_4, E, S, T) \leq_P \text{CSP}(\mathbb{A})$

Try to formulate a general theorem covering these particular cases.

Problem 4. Prove that $\text{CSP}(\mathbb{A}), \text{CSP}(\mathbb{B})$ and $\text{CSP}(\mathbb{C})$ are polynomially equivalent, where

$$\mathbb{A} = (\{0, 1, 2\}, C_0, C_1, C_2, N), \quad N = \{0, 1, 2\}^2 \setminus \{(0, 0), (1, 1), (2, 2)\}$$

$$\mathbb{B} = (\{0, 1\}, S_{000}, S_{001}, S_{011}, S_{111}), \quad S_{ijk} = \{0, 1\}^3 \setminus \{(i, j, k)\}$$

$$\mathbb{C} = (\{0, 1\}, C_0, C_1, R), \quad R = \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}$$

Try to use the general theorem from Problem 3.

Problem 5. Prove that $\text{CSP}(\mathbb{A}) \sim_P \text{CSP}(\{0, 1, 2\}, N)$, where \mathbb{A}, N are from the previous problem.

Problem 6. Let \mathbb{A} be a finite *core* and let \mathbb{B} be the relational structure formed from \mathbb{A} by adding all the unary relations $C_a = \{a\}$, $a \in A$. Prove that $\text{CSP}(\mathbb{A}) \sim_P \text{CSP}(\mathbb{B})$.

Problem 7. Let \mathbb{A} be a finite relational structure such that $\text{CSP}(\mathbb{A})$ is in P. Prove that there is a polynomial-time algorithm for finding a solution of $\text{CSP}(\mathbb{A})$.