CSP lecture 21/22 winter semester – Problem Set 3

An *n-ary operation* on a set A is a mapping $A^n \to A$. *The n-ary projection onto the i-th coordinate* (on a set A) is the operation π_i^n defined by $\pi_i^n(a_1, \dots, a_n) = a_i$ for any $a_1, \dots, a_n \in A$.

An n-ary operation $f : A^n \to A$ is *compatible* with an m-ary relation $R \subseteq A^m$ if $f(\mathbf{r}_1, \dots, \mathbf{r}_n) \in R$ (operation is applied coordinate-wise) whenever $\mathbf{r}_1, \dots, \mathbf{r}_n \in R$. In other words, for any $m \times n$ matrix whose columns are in R, f applied to the rows of this matrix gives a tuple in R. In such a situation, we also say that R is compatible with f, or R is invariant under f.

An operation $A^n \to A$ is a *polymorphism* of a relational structure $\mathbb{A} = (A; \dots)$ if it is compatible with all the relations in $\mathbb{A}$. The set of all polymorphisms of $\mathbb{A}$ is denoted $\mathrm{Pol}(\mathbb{A})$.

Problem 1. Observe that

- $f : A^n \to A$ is compatible with every singleton unary relation $\{a\}$, $a \in A$, iff $f(a, \dots, a) = a$ for all $a \in A$;
- the constant unary operation $c_a : A \to A$ (defined by $c_a(b) = a$ for any $b \in A$) is compatible with $R \subseteq A^n$ iff R contains the tuple $(a, a, \dots, a)$.

Problem 2. Let A be a set. Prove that f is compatible with every relation on A if and only if f is a projection.

Problem 3. Let $\mathbb{A} = (A; \dots)$ be a relational structure, $f \in \mathrm{Pol}(\mathbb{A})$ a binary polymorphism and $g \in \mathrm{Pol}(\mathbb{A})$ a ternary polymorphism. Then the 4-ary operation h defined by

$$h(a, b, c, d) = g(a, f(c, g(b, b, d)), c), \quad a, b, c, d \in A$$

is a polymorphism of $\mathbb{A}$ as well. Try to formulate a general statement.

Problem 4. Find all unary and binary polymorphisms of the structure $\mathbb{A} = (\{0,1\}; H, C_0, C_1)$ from Problem Set 1 (Problem 2 – HORN-SAT).

Problem 5. Find all unary and binary polymorphisms of the structure

$$\mathbb{A} = (\{0,1\}; \text{all unary and binary relations})$$

from Problem Set 1 (Problem 1 – 2-SAT). Find some nice nontrivial (= not a projection) polymorphism of $\mathbb{A}$.

Problem 6. Find all unary, binary, and ternary polymorphisms of the structure $\mathbb{A} = (\{0,1\}; C_0, C_1, G_1, G_2)$ from Problem Set 1 (Problem 3 – LIN-EQ($\mathbb{Z}_2$)).

A relation $R \subseteq A^m$ is *pp-definable* from $\mathbb{A} = (A; \dots)$ if it can be defined from relations in $\mathbb{A}$ by a pp-formula, that is, a formula which only uses conjunction, equality, and existential quantification. A relational structure $\mathbb{B} = (B; \dots)$ is pp-definable from $\mathbb{A}$ if $A = B$ and each relation in $\mathbb{B}$ is pp-definable from $\mathbb{A}$. We also say that $\mathbb{A}$ pp-defines $\mathbb{B}$.

Problem 7. Prove that any relation pp-definable from $\mathbb{A}$ is invariant under every polymorphism of $\mathbb{A}$.

Problem 8. Find all polymorphisms of the structure $\mathbb{B}$ in Problem Set 2, Problem 4. (3–SAT). Hint: only projections; possible approach: (1) pp-define the four-ary relations of the form $R_{a,b,c,d} = \{0,1\}^4 \setminus \{(a,b,c,d)\}$, (2) pp-define all four-ary relations (3) similarly, pp-define every relation, (4) use the results of other problems in this problem set.

Problem 9. Let $\mathbb{A}$ be a finite structure. Prove that a relation invariant under every polymorphism of $\mathbb{A}$ is pp-definable from $\mathbb{A}$. Proof strategy:

(i) Denote $R = \{(c_{11}, \dots, c_{1k}), \dots, (c_{m1}, \dots, c_{mk})\}$

(ii) Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a complete list of m-tuples of elements of A (ie. $n = |A|^m$)

(iii) Prove that the relation

$$S = \{(f(\mathbf{a}_1), \ldots, f(\mathbf{a}_n)) : f \text{ is an } m\text{-ary polymorphism}\}$$

is pp-definable from $\mathbb{A}$ (no need to use existential quantification)

(iv) Existentially quantify over all coordinates but those corresponding to $(c_{11}, \ldots, c_{m1})$, $\ldots$, $(c_{1k}, \ldots, c_{mk})$

(v) Prove that the obtained relation contains R (because of projections) and is contained in R (because of compatibility)

Problem 9'. Let $\mathbb{A} = (\mathbb{Z} \times \mathbb{Z}; R, U)$, where

$$R = \{((x, y), (x', y')) \mid x = x', |y' - y| \in \{1, 2\}\}, \quad U = \{(0, 0)\}.$$

Prove that $\{(0, y) \mid y \in \mathbb{Z}\}$ is invariant under every polymorphism of $\mathbb{A}$, but that this set is not pp-definable from $\mathbb{A}$.

Problem 10. Observe that, for finite structures $\mathbb{A}$ and $\mathbb{B}$,

- $\mathbb{A}$ pp-defines $\mathbb{B}$ iff $\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B})$ and in such a case $\mathrm{CSP}(\mathbb{B}) \leq_P \mathrm{CSP}(\mathbb{A})$;
- any CSP over a two–element structure is polynomially reducible to 3–SAT
- if $\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B})$, then the proof of Problem 9 gives an explicit pp-formulas defining relations in $\mathbb{B}$ from relations in $\mathbb{A}$.
- In particular, for $\mathbb{B}$ and $\mathbb{C}$ as in Problem Set 2, Problem 4, we get $\mathrm{CSP}(\mathbb{C}) \leq \mathrm{CSP}(\mathbb{B})$. How large are the explicit formulas defining relations in $\mathbb{C}$ from relations in $\mathbb{B}$?