CSP lecture 21/22 winter semester – Problem Set 4

A set of operations on a set A is a *(function) clone* on A if it contains all projections and is closed under composition (as in Problem 3, Problem Set 3). A function clone on A is called *idempotent* if for every operation f in it and every $a \in A$, $f(a, a, \dots, a) = a$.

Problem 0. Recall that for any relational structure $\mathbb{A}$, $\mathrm{Pol}(\mathbb{A})$ is a clone.

In this problem set, we focus on function clones on the set $A = \{0, 1\}$. We use the following notation for some special operations on $\{0, 1\}$:

$\wedge$ the binary minimum operation

$\vee$ the binary maximum operation

maj the ternary majority operation defined by $maj(a, a, b) = maj(a, b, a) = maj(b, a, a) := a$ for every $a, b \in \{0, 1\}$

min the ternary minority operation defined by $min(a, a, b) = min(a, b, a) = min(b, a, a) := b$ for every $a, b \in \{0, 1\}$

An operation $f : A^n \to A$ is called *essentially unary* if there exist i and a unary operation $\alpha : A \to A$ such that $f(x_1, \dots, x_n) = \alpha(x_i)$ for every $x_1, \dots, x_n \in A$.

Problem 1. Assume $\mathcal{A}$ is an idempotent clone on $A = \{0, 1\}$ that contains neither $\wedge$ nor $\vee$. Show that the only binary operations are the two projections.

Problem 2. Assume $\mathcal{A}$ is an idempotent clone on $A = \{0, 1\}$ that contains neither of the operations $\wedge, \vee, maj, min$. Show that the only binary and ternary operations are the projections.

Problem 3. Assume $\mathcal{A}$ is an idempotent clone on $A = \{0, 1\}$ that contains neither of the operations $\wedge, \vee, maj, min$. Show that $\mathcal{A}$ contains only projections. Possible strategy:

- Let $f \in \mathcal{A}$ be n-ary with $n \geq 4$.
- Assume first $f(1, 0, 0, \dots, 0) = 1$. Use the binary operation $g(x, y) := f(x, y, \dots, y)$ to show that $f(0, 1, \dots, 1) = 0$. Use ternary operations of the form $g(x, y, z) := f(w_1, w_2, \dots)$ where $w_1, w_2, \dots \in \{x, y, z\}$ to show that f is the projection onto the first coordinate.
- Deduce that if f is not a projection, then $f(x, \dots, x, y, x, \dots, x) = x$ for every x, y and every position of y.
- Assuming this and using appropriate ternary operations (similar as above) show that $f(x, \dots, x, y, y) = x$, …, etc, and derive a contradiction

Problem 4. Let $\mathcal{A}$ be a clone on $A = \{0, 1\}$ with an operation which is not essentially unary. Prove that $\mathcal{A}$ contains a constant unary operation, or at least one of the operations $\wedge, \vee, maj, min$. Hint: try to reduce to the idempotent case.