Universal Algebra Exercises - Sheet 2

Definition. A lattice L is called *distributive*, if for all $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{2.1}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{2.2}$$

Exercise 9. Show that every lattice with less than four elements is distributive. Find examples of distributive lattices with a large number of elements.

Exercise 10. Show that in the definition of distributive lattices (3.1) and (3.2) are equivalent.

Definition. A lattice L is called *modular*, if for all $x, y, z \in L$

$$x \leq z \implies x \vee (y \wedge z) = (x \vee y) \wedge z$$

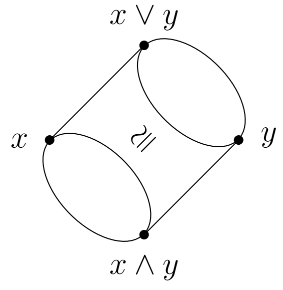
Exercise 11. Show that every distributive lattice is modular and disprove the converse, i.e. find a modular lattice that is not distributive.

Exercise 12. Show that the following two statements hold for all lattices L and all $x, y, z \in L$

$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$$

$$x \leq z \implies x \vee (y \wedge z) \leq (x \vee y) \wedge z$$

Exercise 13 (Diamond isomorphism theorem)**.** Let L be a modular lattice and $x, y \in L$. Show that the intervals $I[x \wedge y, x]$ and $I[y, x \vee y]$ are isomorphic lattices.

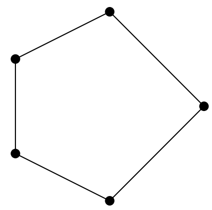


Exercise 14. A term $m(x, y, z)$ of an algebra A is called *majority* if it satisfies the identities

$$x \approx m(x, x, y) \approx m(x, y, x) \approx m(y, x, x)$$

Show that every lattice has a majority term.

Theorem (Dedekind)**.** Prove that a lattice is modular if and only if it does not contain the following lattice as a sublattice.



Theorem (Birkhoff)**.** A modular lattice is distributive if and only if it does not contain the following lattice as a sublattice.