Universal Algebra Exercises - Sheet 3

Exercise 15. Show that every complete lattice is bounded.

Exercise 16. Find examples of lattices L that contain a sublattice S such that

(i) L is complete but S is not complete

(ii) L is not complete but S is complete

(iii) both L and S are complete lattices but S is not a complete sublattice

Exercise 17. Let L be a complete lattice and $a, b \in L$ two compact elements.

(i) Is $a \vee b$ compact?

(ii) Is $a \wedge b$ compact?

Exercise 18. Let C be a closure operator on a set X. Prove that L_C is closed under finite unions if and only if for all subsets $U, V \in 2^X$

$$C(U \cup V) = C(U) \cup C(V)$$

Exercise 19. Let X be a set and let ϕ be the binary relation on 2^X defined by

$$(U, V) \in \phi \iff U \cap V \neq \emptyset$$

Consider the Galois correspondence on the sets 2^{2^X} and 2^{2^X} induced by this relation

(i) Let $X = \{1, 2, 3, 4\}$. Compute $A^{\leftarrow\rightarrow}$ and $A^{\rightarrow\leftarrow}$ for both $A = \Big\{\{1, 2\}, \{2, 3\}\Big\}$ and $A = \Big\{\{1, 2\}, \{2\}\Big\}$. Compare the results.

(ii) Prove that if a Galois correspondence is defined by a symmetric relation on a set, then the closure operators induced by it coincide.

(iii) Prove that for every $A \subseteq 2^X$ we have

$$A^{\rightarrow\leftarrow} = \{U \in 2^X \mid \exists V \in A, V \subseteq U\}$$

Exercise 20. Let C be a closure operator on a set X. Find a relation $\phi \subseteq X \times 2^X$ whose induced Galois correspondence gives

$$C(U) = U^{\rightarrow\leftarrow}$$

for all subsets $U \subseteq X$.