Universal Algebra Exercises - Sheet 4

Exercise 19. Let X be a set and let ϕ be the binary relation on 2^X defined by

$$(U, V) \in \phi \iff U \cap V \neq \emptyset$$

Consider the Galois correspondence on the sets 2^{2^X} and 2^{2^X} induced by this relation

(i) Let $X = \{1, 2, 3, 4\}$. Compute $A^{\leftarrow\rightarrow}$ and $A^{\rightarrow\leftarrow}$ for both $A = \Big\{\{1, 2\}, \{2, 3\}\Big\}$ and $A = \Big\{\{1, 2\}, \{2\}\Big\}$. Compare the results.

(ii) Prove that if a Galois correspondence is defined by a symmetric relation on a set, then the closure operators induced by it coincide.

(iii) Prove that for every $A \subseteq 2^X$ we have

$$A^{\rightarrow\leftarrow} = \{U \in 2^X \mid \exists V \in A, V \subseteq U\}$$

Exercise 20. Let C be a closure operator on a set X. Find a relation $\phi \subseteq X \times 2^X$ whose induced Galois correspondence gives

$$C(U) = U^{\rightarrow\leftarrow}$$

for all subsets $U \subseteq X$.

Exercise 21. Let $\mathbb{A} = (A, *)$ be a binary algebra and θ an equivalence relation on A. Show that θ is a congruence relation if and only if for all $a, b, c \in A$ we have

$$(a, b) \in \theta \implies \begin{cases} (a * c, b * c) \in \theta & \text{and} \\ (c * a, c * b) \in \theta \end{cases}$$

Exercise 22. Let $\mathbb{A} = (A, *)$ be an algebra where $A = \{0, 1, 2, 3\}$ and $*$ is defined by the following multiplication table.

$*$	0	1	2	3
0	0	2	1	1
1	2	1	0	2
2	1	0	2	0
3	1	2	0	3

Draw the lattice of subalgebras and the lattice of congruences of $\mathbb{A}$.

Exercise 23. Consider the algebra $(\mathbb{Z}, +, \cdot) \times (\mathbb{Z}, \cdot, +)$. What is the subalgebra generated by the pairs $(0, 1)$ and $(1, 0)$?

Exercise 24. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras in the same signature and let $f : \mathbb{A} \to \mathbb{B}$ be a homomorphism.

- Given two subalgebras $U \leq \mathbb{A}$ and $V \leq B$, are $f(U) \subseteq \mathbb{B}$ and $f^{-1}(V) \subseteq \mathbb{A}$ subalgebras?
- Given two congruences $\theta \in \mathrm{Con}(\mathbb{A})$ and $\psi \in \mathrm{Con}(\mathbb{B})$, is $f(\theta) \in \mathrm{Con}(\mathbb{B})$ and $f^{-1}(\psi) \in \mathrm{Con}(\mathbb{A})$?
- Given a subset $X \subseteq A$ is $f(\mathrm{Sg}_{\mathbb{A}}(X)) = \mathrm{Sg}_{\mathbb{B}}(f(X))$?

Exercise 25. Given a binary algebra $\mathbb{A} = (A, *)$ define its *nucleus* as

$$B := \{a \in A \mid \forall x, y \in A, (x * a) * y = x * (a * y)\}$$

Show that B is a subalgebra of $\mathbb{A}$ and find and an example of an algebra $\mathbb{A}$ whose nucleus is empty.