Universal Algebra Exercises - Sheet 5

Exercise 24. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras in the same signature and let $f : \mathbb{A} \to \mathbb{B}$ be a homomorphism.

- Given two subalgebras $U \leq \mathbb{A}$ and $V \leq B$, are $f(U) \subseteq \mathbb{B}$ and $f^{-1}(V) \subseteq \mathbb{A}$ subalgebras?
- Given two congruences $\theta \in \mathrm{Con}(\mathbb{A})$ and $\psi \in \mathrm{Con}(\mathbb{B})$, is $f(\theta) \in \mathrm{Con}(\mathbb{B})$ and $f^{-1}(\psi) \in \mathrm{Con}(\mathbb{A})$?
- Given a subset $X \subseteq A$ is $f(\mathrm{Sg}_{\mathbb{A}}(X)) = \mathrm{Sg}_{\mathbb{B}}(f(X))$?

Exercise 26. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras of the same type and let $f : A \to B$ be a map. Show that f is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ if and only if its graph is a subalgebra of $\mathbb{A} \times \mathbb{B}$.

$$\{(a, f(a)) \mid a \in A\} \leq \mathbb{A} \times \mathbb{B}$$

Exercise 27. Let $f, g : \mathbb{A} \to \mathbb{B}$ be two homomorphisms and let $X \subseteq A$ with $\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}(X)$. Show that

$$f|_X = g|_X \implies f = g.$$

Remark. The converse is also true: Given $\mathbb{A}$ and $X \subseteq A$, then $\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}(X)$ if and only if $f|_X = g|_X$ implies $f = g$ for all algebras $\mathbb{B}$ and all homomorphisms $f, g : \mathbb{A} \to \mathbb{B}$.

Exercise 28. Let $X \subseteq A$ be a subset that generates the algebra $\mathbb{A}$ such that no proper subset of X generates $\mathbb{A}$. Is it true that every map $f : X \to B$ to any algebra $\mathbb{B}$ can be extended to an homomorphism $\mathbb{A} \to \mathbb{B}$?

Exercise 29. Find all homomorphisms $(\mathbb{N}, +)^2 \to (\mathbb{Z}_2, +)$.

Exercise 30. Show that a map $f : A \to B$ is injective if and only if its kernel is the equality relation.

Exercise 31 (Second isomorphism theorem)**.** Let $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{C}$ be two homomorphisms and let $\alpha \leq \beta$ be two congruences on $\mathbb{A}$ and let ϕ be a congruence on $\mathbb{B}$. Prove that

(i) if f is surjective and $\ker(f) \subseteq \ker(g)$, then there exists a homomorphism $h : \mathbb{B} \to \mathbb{C}$ such that $g = h \circ f$.

$$\begin{tikzcd} \mathbb{A} \arrow[r, two heads, "f"] \arrow[dr, "g"'] & \mathbb{B} \arrow[d, "\exists h"] \\ & \mathbb{C} \end{tikzcd}$$

(ii) there is an embedding $\mathbb{A}/f^{-1}(\phi) \to \mathbb{B}/\phi$.

(iii) there is a congruence β/α on $\mathbb{A}/\alpha$ such that

$$\mathbb{A}/\beta = (\mathbb{A}/\alpha)\big/(\beta/\alpha).$$

Exercise 32. Find classes of algebras witnessing that

$$\mathrm{PS} \lneq \mathrm{SP} \qquad \mathrm{PH} \lneq \mathrm{HP} \qquad \mathrm{SH} \lneq \mathrm{HS}$$