Universal Algebra Exercises - Sheet 6

Exercise 31 (Second isomorphism theorem)**.** Let $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{C}$ be two homomorphisms and let $\alpha \leq \beta$ be two congruences on $\mathbb{A}$ and let ϕ be a congruence on $\mathbb{B}$. Prove that

(i) if f is surjective and $\ker(f) \subseteq \ker(g)$, then there exists a homomorphism $h : \mathbb{B} \to \mathbb{C}$ such that $g = h \circ f$.

$$\begin{tikzcd} \mathbb{A} \arrow[r, two heads, "f"] \arrow[dr, "g"'] & \mathbb{B} \arrow[d, "\exists h"] \\ & \mathbb{C} \end{tikzcd}$$

(ii) there is an embedding $\mathbb{A}/f^{-1}(\phi) \to \mathbb{B}/\phi$.

(iii) there is a congruence β/α on $\mathbb{A}/\alpha$ such that

$$\mathbb{A}/\beta = (\mathbb{A}/\alpha)\Big/(\beta/\alpha).$$

Exercise 33. Consider the algebra $C_n = (\{0, 1, \ldots, n-1\}, f)$, where f is the unary function $x \mapsto x + 1 \mod n$. Decide for each $n \in \{2, 3, 4, 5, 6\}$ if C_n is simple, subdirectly irreducible or directly indecomposable.

Exercise 34. Consider the algebra $\mathbb{A} = (\{0, 1, 2, 3, 4\}, g)$, where g is the unary function given by the following diagram.

$$\begin{tikzcd} & & & 4 \arrow[dl] & \\ 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[rr] & & 3 \arrow[ul] \end{tikzcd}$$

Draw the congruence lattice of $\mathbb{A}$ and then decide whether $\mathbb{A}$ is subdirectly irreducible and or directly indecomposable.

Exercise 35. Consider the to algebras $\mathbb{A} := (\{0, 1\}, +, \mathrm{id})$ and $\mathbb{B} = (\{0, 1\}, +, f)$, where $+$ is addition modulo 2 and f is the unary function given by $x \mapsto x + 1$ mod 2.

(i) Show that $d : \mathbb{B}^2 \to \mathbb{A}, (x, y) \mapsto x + y$ is a surjective homomorphism.

(ii) Show that the congruence lattice of $\mathbb{B}^2$ is

$$
\begin{tikzcd}
 & 1 \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
\ker(\pi_1) \arrow[dr, no head] & \ker(d) \arrow[d, no head] & \ker(\pi_2) \arrow[dl, no head] \\
 & 0 &
\end{tikzcd}
$$

(iii) Show that $\mathbb{B}^2 \cong \mathbb{B} \times \mathbb{A} \ncong \mathbb{A}^2$ and conclude that direct decompositions are in general not unique.

Exercise 36. Find algebras $\mathbb{A}$ and $\mathbb{B}$ such that there are no homomorphisms

$$\mathbb{A} \to \mathbb{A} \times \mathbb{B} \quad \text{and} \quad \mathbb{B} \to \mathbb{A} \times \mathbb{B}$$

Exercise 37. Find a proper subdirect composition of the three element lattice into subdirectly irreducible lattices.