

Universal Algebra Exercises - Sheet 6

Exercise 31 (Second isomorphism theorem). Let $f : \mathbb{A} \rightarrow \mathbb{B}$ and $g : \mathbb{A} \rightarrow \mathbb{C}$ be two homomorphisms and let $\alpha \leq \beta$ be two congruences on $\mathbb{A}$ and let ϕ be a congruence on $\mathbb{B}$. Prove that

- (i) if f is surjective and $\ker(f) \subseteq \ker(g)$, then there exists a homomorphism $h : \mathbb{B} \rightarrow \mathbb{C}$ such that $g = h \circ f$.

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{f} & \mathbb{B} \\ & \searrow g & \downarrow \exists h \\ & & \mathbb{C} \end{array}$$

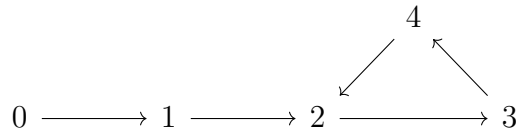
- (ii) there is an embedding $\mathbb{A}/f^{-1}(\phi) \rightarrow \mathbb{B}/\phi$.

- (iii) there is a congruence β/α on $\mathbb{A}/\alpha$ such that

$$\mathbb{A}/\beta = (\mathbb{A}/\alpha)/(\beta/\alpha).$$

Exercise 33. Consider the algebra $C_n = (\{0, 1, \dots, n-1\}, f)$, where f is the unary function $x \mapsto x + 1 \pmod n$. Decide for each $n \in \{2, 3, 4, 5, 6\}$ if C_n is simple, subdirectly irreducible or directly indecomposable.

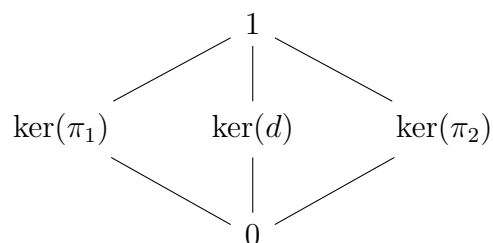
Exercise 34. Consider the algebra $\mathbb{A} = (\{0, 1, 2, 3, 4\}, g)$, where g is the unary function given by the following diagram.



Draw the congruence lattice of $\mathbb{A}$ and then decide whether $\mathbb{A}$ is subdirectly irreducible and or directly indecomposable.

Exercise 35. Consider the two algebras $\mathbb{A} := (\{0, 1\}, +, \text{id})$ and $\mathbb{B} = (\{0, 1\}, +, f)$, where $+$ is addition modulo 2 and f is the unary function given by $x \mapsto x + 1 \pmod 2$.

- (i) Show that $d : \mathbb{B}^2 \rightarrow \mathbb{A}, (x, y) \mapsto x + y$ is a surjective homomorphism.
- (ii) Show that the congruence lattice of $\mathbb{B}^2$ is



- (iii) Show that $\mathbb{B}^2 \cong \mathbb{B} \times \mathbb{A} \not\cong \mathbb{A}^2$ and conclude that direct decompositions are in general not unique.

Exercise 36. Find algebras $\mathbb{A}$ and $\mathbb{B}$ such that there are no homomorphisms

$$\mathbb{A} \rightarrow \mathbb{A} \times \mathbb{B} \quad \text{and} \quad \mathbb{B} \rightarrow \mathbb{A} \times \mathbb{B}$$

Exercise 37. Find a proper subdirect composition of the three element lattice into subdirectly irreducible lattices.