Universal Algebra Exercises - Homework 1

Exercise 1. A *latin square* is an algebra $(A, *)$, where $*$ is a binary operation such that for all $a, b \in A$ there exists a unique $x \in A$ such at $x * a = b$ and a unique $y \in A$ such that $a * y = b$, which we denote by b/a and $a\backslash b$ respectively. A *quasigroup* is an algebra $(A, *, /, \backslash)$ with three binary operations, satisfying the following identities.

$$y \approx x * (x\backslash y) \approx x\backslash(x * y) \approx (y/x) * x \approx (y * x)/x$$

Let Φ be the map assigning to every latin square a quasi group as above, and let Ψ the map assigning to every quasi group a latin square by forgetting the operations $/$ and $\backslash$. Show that Φ and Ψ are inverse to each other. Also, given two quasigroups $(A, *, /, \backslash)$ and $(A', *', /', \backslash')$, show that a map $f : A \to A'$ is a homomorphism of quasigroups if and only if it is a homomorphism of the corresponding latin squares.

Exercise 2. A set $C \subseteq \mathbb{R}$ is called *convex* if $x, y \in C$ implies $\theta x + (1-\theta)y \in C$ for all $\theta \in [0, 1]$. Let $\mathrm{Cvx}(\mathbb{R}^n)$ be the set of all convex subsets of $\mathbb{R}^n$. Show that $(\mathrm{Cvx}(\mathbb{R}^n), \subseteq)$ is a complete lattice.

Exercise 3. A map $f : L_1 \to L_2$ between two lattices is called *monotone* if $x \leq y$ implies $f(x) \leq f(y)$ for all elements $x, y \in L_1$. Given a complete lattice L and a monotone map $f : L \to L$, show that there is a point $a \in L$ with $f(a) = a$.