Universal Algebra Exercises - Homework 2

Exercise 1. Determine all subalgebras and congruences of $(\mathbb{N}, +, *)$, where

$$x * y = \begin{cases} 0 & \text{if } y = 0, 1 \\ x \mod y & \text{otherwise} \end{cases}$$

and draw the lattices $\mathrm{Sub}(\mathbb{N}, +, *)$ and $\mathrm{Con}(\mathbb{N}, +, *)$.

Exercise 2. Given a group G, prove that the lattice of its normal subgroups is isomorphic to its lattice of congruences.

Exercise 3. Fix a prime number p and consider the algebra $\mathbb{A} := (\{0, 1, \ldots, p-1\}, m)$, where m is the ternary operation defined by

$$m(x, y, z) = x - y + z \mod p.$$

Prove that for any n, $R \subseteq A^n$ is a subalgebra of $\mathbb{A}^n$ if and only if R is empty or an affine subspace of the vector space $\mathbb{Z}_p^n$ (recall affine subspaces from linear algebra).