Universal Algebra 2 - Exercises 3

Exercise 3.1. Let $\mathbb{L} = (L, \wedge, \vee)$ be a distributive lattice. Show that

$$(c, d) \in Cg_{\mathbb{L}}((a, b)) \iff \begin{cases} c \wedge (a \wedge b) & = d \wedge (a \wedge b) \\ c \vee (a \vee b) & = d \vee (a \vee b) \end{cases}$$

Hence the variety of distributive lattices has DPC.

Exercise 3.2. Show that the variety of semilattices has DPC.

Exercise 3.3. Recall that a variety is already finitely based, if it has definable principal congruences and finitely many subdirectly irreducibles (up to isomorphism). Consider commutative rings R that satisfy equation $x^n \approx x$.

- Show that every subdirecly irreducible such ring is a field of order d, where $d - 1 \mid n - 1$.
- Conclude that $\mathrm{HSP}(R)$ is finitely based.

Exercise 3.4. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras of finite type on the same domain with $\mathrm{Clo}(\mathbb{A}) = \mathrm{Clo}(\mathbb{B})$. Show that $\mathbb{A}$ is finitely based if and only if $\mathbb{B}$ is finitely based. Is this still true if we do not assume finite type?

Exercise 3.5. Let $\mathbb{A}$ be a finite relational structure. Show that there exists $\mathbb{B}$ such that:

- there are homomorphisms $\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$
- $\mathrm{End}(\mathbb{B}) = \mathrm{Aut}(\mathbb{B})$

Further, show that $\mathbb{B}$ is unique up to isomorphism. ($\mathbb{B}$ is called the core of $\mathbb{A}$).