Universal Algebra 2 - Exercises 4

Exercise 4.1. Let $\mathbb{A}$ be a finite idempotent algebra. Show that if there is a naked set in $\mathrm{HSP}(\mathbb{A})$, then there is a naked set in $\mathrm{HS}(\mathbb{A})$.

- Recall that, since $\mathbb{A}$ is finite, any finite algebra $\mathbb{B} \in \mathrm{HSP}(\mathbb{A})$ is also in $\mathrm{HSP}_{\text{fin}}$.
- If

$$\begin{aligned} A &= \{1, 2, 3, 4, \dots\} \\ R &= \{11, 12, 13, 14, \dots\} \leq \mathbb{A} \\ \sim &= \dots 11, 12, 34 \mid 13 \cdots \in \mathrm{Con}(R) \end{aligned}$$

  and $R/\sim$ naked, find a naked set in $\mathrm{HS}(\mathbb{A})$
- If

$$\begin{aligned} A &= \{1, 2, 3\} \\ R &= \{13, 23, 21, 32, \dots\} \leq \mathbb{A} \\ \sim &= \dots 13, 23, 21 \mid 32 \cdots \in \mathrm{Con}(R) \end{aligned}$$

  and $R/\sim$ naked, find a naked set in $\mathrm{HS}(\mathbb{A})$
- Prove the implication for HSP_2, then generalize.

Exercise 4.2. Take $\mathbb{A} = (\{0, 1, \dots, 4\}; f)$ where $f(i) := i + 1 \pmod 5$.

- What is $\mathrm{Clo}(\mathbb{A})$?
- Prove that there is a naked set in $\mathrm{HSP}(\mathbb{A})$.
- **bonus** Explicitly construct the naked set.
- Prove that there is no naked set in $\mathrm{HS}(\mathbb{A})$.

- Find a simpler proof that for finite $\mathbb{A}$:

$$\text{naked}_2 \in \text{HSP}(\mathbb{A}) \not\Rightarrow \text{naked}_2 \in \text{HS}(A)$$

Exercise 4.3. Take $\mathbb{A} := (\mathbb{N}; \text{all injective operations})$.

- What is $\mathcal{A} \coloneqq \text{Clo}(\mathbb{A})$?

- Prove that $\mathcal{A}$ contains no Taylor operation.

- Show that the are $f, u \in \mathcal{A}$ such that $f(y,x) \approx u(f(x,y))$, so, in particular, $\mathcal{A} \overset{\text{clone}}{\not\to} \text{Proj} \not\Rightarrow A$ contains Taylor.

- **bonus** Find non-trivial height 1 identities satisfied by $\mathcal{A}$, so, in particular $\mathcal{A} \overset{\text{minion}}{\not\to} \text{Proj} \not\Rightarrow$ contains Taylor.

Exercise 4.4. Prove that if $\mathcal{A}$ contains a p-ary cyclic operation for every sufficiently large prime p, then a has a 4-ary Siggers operation $s(r,a,r,e) \approx s(a,r,e,a)$. (Hint: identify variables in cyclic operation of suitable arity)

Exercise 4.5. Let $(0,1)$ be the real unit interval and take $\mathcal{A}$ to be the set of functions $f\colon (0,1)^n \to (0,1)$ which are idempotent and act like a projection in the limit to $\{0,1\}^n$.

- Formalize this definition and show that $\mathcal{A}$ is a clone.

- Find a clone homomorphism $\mathcal{A} \to \text{Proj}$ (hence there is a naked set in $\text{HSP}(\mathcal{A})$).

- Prove that there is no naked set in $\text{HS}(\mathcal{A})$.

- Compare this result to Exercise 4.1.