Universal Algebra Exercises - Homework 1

Exercise 1.1. Prove that a loop $(L, \cdot, /, \backslash, 1)$ is abelian if and only if $\cdot$ is a commutative group operation.

Exercise 1.2. Consider the loop L with universe $\mathbb{Z}_4 \times \mathbb{Z}_2$ and the multiplication

$$(a,b) \cdot (c,d) := \begin{cases} (a * c, 0) & \text{if } b = d = 1 \\ (a + c, b + d) & \text{otherwise} \end{cases}$$

where $*$ is defined as follows.

$*$	0	1	2	3
0	1	0	2	3
1	0	2	3	1
2	2	3	1	0
3	3	1	0	2

Consider the map $f : L \to \mathbb{Z}_2$, $(a, b) \mapsto b$ and its kernel α.

- Prove that f is a homomorphism.
- Prove that α is not an *abelian congruence*, i.e. $C(\alpha, \alpha; 0)$ does not hold.

Exercise 1.3. Consider an algebra $\mathbb{A}$ and the congruence $\alpha := [1_{\mathbb{A}}, 1_{\mathbb{A}}]$.

- Show that α is the smallest congruence such that $\mathbb{A}/\alpha$ is abelian.
- Conclude that for any abelian algebra $\mathbb{B}$ (in the same signature as $\mathbb{A}$) there is a natural[1] bijection between the following sets of homomorphisms.
$$\hom(\mathbb{A}/\alpha, \mathbb{B}) \cong \hom(\mathbb{A}, \mathbb{B})$$
- (Bonus) Assume that $\mathbb{C}$ is an abelian algebra such that for any abelian algebra $\mathbb{B}$ there is a natural bijection $\hom(\mathbb{C}, \mathbb{B}) \cong \hom(\mathbb{A}, \mathbb{B})$. Show that $\mathbb{C} \cong \mathbb{A}/\alpha$.

[1]Ignore this word if you don't know what is means.