Universal Algebra Exercises - Homework 3

Let t be an n-ary operation on A, B a set, $f\colon A \to B$ and $g\colon B \to A$ mappings. We define an n-ary operation $t^{(f,g)}$ (a *reflection* of t) on B by

$$t^{(f,g)}(x_1, \ldots, x_n) := f(t(g(x_1), \ldots, g(x_n)))$$

For clones $\mathcal{A}, \mathcal{B}$ write $\mathcal{B} \in \mathrm{ER}(\mathcal{A})$ if there exist $f\colon A \to B$ and $g\colon B \to A$ such that

$$\{t^{(f,g)} : t \in \mathcal{A}\} \subseteq \mathcal{B}.$$

Exercise 3.1. Prove that the following are equivalent for any clones $\mathcal{A}, \mathcal{B}$ on finite sets:

(i) $\mathcal{A} \leq^{\min} \mathcal{B}$

(ii) $\mathcal{B} \in \mathrm{ERP}(\mathcal{A})$

(Hint: use an argument similar to the proof of HSP)

Exercise 3.2. A clone $\mathcal{A}$ on a finite set A is called a *core* if each unary member of $\mathcal{A}$ is a bijection.

a) Prove that for any core clone $\mathcal{A}$ on a finite set and a unary member $f \in \mathcal{A}$, $f^{-1} \in \mathcal{A}$.

b) Prove that for any clone $\mathcal{A}$ on a finite set there is a core clone $\mathcal{B}$ on a finite set such that $\mathcal{A} \sim^{\min} \mathcal{B}$.

(Hint for b): Consider the image S of a unary $f \in \mathcal{A}$ (suitably chosen) and define $\mathcal{B}$ on S as those restrictions of members of $\mathcal{A}$ that preserve S.)

Exercise 3.3. An operation f is called *essentially injective* if it is a minor of an injective function, say $f(x_1, \ldots, x_n) = g(x_{i_1}, \ldots, x_{i_k})$ with g injective. Let $\mathcal{A}$ be the set of all essentially injective operations on the set $\mathbb{N}$.

- Show that $\mathcal{A}$ is a clone.
- Show that $\mathcal{A}$ does not contain a Taylor operation.

- However, show that there are $f, u \in \mathcal{A}$ such that $f(x, y) = u \circ f(y, x)$. Conclude that there is no clone homomorphism from $\mathcal{A} \to \text{Proj}$.

Remark. There is also no minion homomorphism from $\mathcal{A} \to \text{Proj}$, but that is harder to show. Note that this exercise highlights the importance of "idempotence" in Taylors Theorem.