NMAG 455 Universal Algebra 2, spring semester 2017–2018

Homework 3

Deadline 9 May 2018, 9:00

Let t be an n-ary operation on A, B a set, and $f : A \to B$, $g : B \to A$ mappings. We define an n-ary operation $t^{(f,g)}$ on B by

$$t^{(f,g)}(x_1, \dots, x_n) = f(t(g(x_1), \dots, g(x_n))) \ .$$

(this is called a *reflection* of t)

For clones $\mathcal{A}, \mathcal{B}$, write $\mathcal{B} \in ER(\mathcal{A})$ if there are $f : A \to B$, $g : B \to A$ such that

$$\{t^{(f,g)} : t \in \mathcal{A}\} \subseteq \mathcal{B} \ .$$

3.1. (10 points) Prove that the following are equivalent for any clones $\mathcal{A}, \mathcal{B}$ on finite sets:

(i) $\mathcal{A} \overset{h1}{\leq} \mathcal{B}$

(ii) $\mathcal{B} \in ERP(\mathcal{A})$ (ie. $\mathcal{B} \in ER(\mathcal{C})$ for some power $\mathcal{C}$ of $\mathcal{A}$)

(Hint: use an argument similar to the proof of the analogous theorem for $\leq$.)

A clone $\mathcal{A}$ on a finite set A is called a *core* if each unary member of $\mathcal{A}$ is a bijection.

3.2 (10 points)

(i) Prove that for any core clone $\mathcal{A}$ on a finite set and a unary member f, $f^{-1} \in \mathcal{A}$.

(ii) Prove that for any clone $\mathcal{A}$ on a finite set there is a core clone $\mathcal{B}$ on a finite set such that $\mathcal{A} \overset{h1}{\sim} \mathcal{B}$

(Hint for (ii): use a non-bijective unary member of $\mathcal{A}$ to define a $\overset{h1}{\sim}$ equivalent clone on a smaller set.)

3.3 (10 points) Prove that for any core clone $\mathcal{A}$, $\mathcal{A} \overset{h1}{\sim} \mathcal{A}^{id}$, where

$$\mathcal{A}^{id} = \{f \in \mathcal{A} : f \text{ is idempotent } \}$$

(Modify each operation in $\mathcal{A}$ (by composing it with a unary operation) to make it idempotent and prove that this modification is an h1-homomorphism.)