NMAG 405 Universal Algebra 1, fall semester 2017–2018

Homework 3

Deadline 7 Dec 2017, 10:40

3.1. (10 points) Let $\mathbf{A}$ be an algebra, n a positive integer, and $\theta_1, \dots, \theta_n \in \operatorname{Con} \mathbf{A}$. Suppose that $\bigcap_{i=1}^{n} \theta_i = 0$ and, for every $i \leq n$, $\theta_i \circ \bigcap_{j \neq i} \theta_j = 1$. Prove that $\mathbf{A} \cong \prod_{i=1}^{n} \mathbf{A}/\theta_i$. Also prove that every direct decomposition into finitely many factors arises in this way.

3.2. (10 points) Let $\mathbf{R}$ be a commutative ring with identity, and assume that $\mathbf{R}$ has no nilpotent elements.

(a) Let a be a nonzero element of $\mathbf{R}$. Prove that $\mathbf{R}$ has a prime ideal P_a with $a \notin P_A$. (Hint: Find an ideal maximal with respect to the exclusion of a, a^2, a^3, $\dots$)

(b) Prove that $\mathbf{R}$ is a subdirect product of integral domains.

3.3. (10 points) Let $\mathbf{L}, \mathbf{M}$ be finite non-trivial lattices (i.e., more than one element). Define $\mathbf{L} \oplus \mathbf{M}$ to be the lattice with universe $L \dot{\cup} M$ and ordered so that every element of $\mathbf{L}$ lies below every element of $\mathbf{M}$. Prove that $\mathrm{HSP}(\{\mathbf{L}, \mathbf{M}\}) = \mathrm{HSP}(\{\mathbf{L} \oplus \mathbf{M}\})$. (Hint: subdirect representation.)