NMAG 405 - Universal Algebra 1 - fall semester 2020/21

Homework 3

Deadline 03.12.2020, 9:00

1. (10 points) Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups. Define $\mathcal{V} \cdot \mathcal{W}$ to be the class of all groups $\mathbf{A}$ containing a normal subgroup $\mathbf{B}$ such that $\mathbf{B} \in \mathcal{V}$ and $\mathbf{A}/\mathbf{B} \in \mathcal{W}$. Prove that $\mathcal{V} \cdot \mathcal{W}$ is a variety.

2. (10 points) Let $\mathbf{L}, \mathbf{M}$ be two bounded lattices with more than one element. By $\mathbf{L} \oplus \mathbf{M}$ we define the lattice with universe $L \dot{\cup} M$ such that every element of L lies below every element on M. Prove that $\mathsf{HSP}(\{\mathbf{L}, \mathbf{M}\}) = \mathsf{HSP}(\{\mathbf{L} \oplus \mathbf{M}\})$

3. (10 points) Let $\mathbf{R}$ be a commutative ring such that for every non-zero element a also $a^n \neq 0$ holds for every $n > 1$ in $\mathbf{R}$ (such a ring is also called *reduced*).

   - Show that for every $a \in R \setminus \{0\}$ there is a prime ideal P_a with $a \notin P_a$ (Hint: Use Zorn's lemma to show that there is a maximal ideal P_a that does not contain $a, a^2, a^3, \ldots$)
   - Prove that $\mathbf{R}$ is the subdirect product of integral domains.