VA 4.1

II SEMANTICS

- basic constructions
- variety
- homomorphisms, isomorphism theorems
- direct & subdirect decompositions

BASIC CONSTRUCTIONS

concern algebras of the same signature / type

H quotients $\doteq$ homomorphic images (up to isomorphism)

S subalgebras

P products

UA $\boxed{4.2}$

S SUBALGEBRAS

$\underline{A} = (A; \underset{\text{0-ary}}{f}, \underset{\text{1-ary}}{g}, \underset{\text{2-ary}}{h})$, $B \subseteq A$

$\rightsquigarrow \underline{B} = (B; \underbrace{f, g\restriction_B, h\restriction_{B^2}}_{\text{must make sense}})$

$\boxed{\text{Def}}$ $\underline{A} = (A; \dots)$ algebra, $B \subseteq A$.

B is a <u>subuniverse</u> of $\underline{A}$ if it is closed under all basic operations of $\underline{A}$ (ie $\forall f$ basic op. of arity n $\forall b_1, \dots, b_n \in B$ $f(b_1, \dots, b_n) \in B$)

f 0-ary	$f \in B$
g 1-ary	$\forall b \in B$ $g(b) \in B$
h 2-ary	$\forall b_1, b_2 \in B$ $h(b_1, b_2) \in B$

<u>Notation</u>: $B \leq \underline{A}$

- can be $\emptyset$

? When $\emptyset \leq \underline{A}$?

$\boxed{\text{Def}}$ $\underline{B}$ is a <u>subalgebra</u> of $\underline{A}$, written $\underline{B} \leq \underline{A}$, if

- $\underline{B}$ has the same signature Σ as $\underline{A}$
- $B \neq \emptyset$
- $\forall f \in \Sigma$ $f^{\underline{B}} = f^{\underline{A}} \restriction B^{ar(f)}$ (necessarily $B \leq \underline{A}$)

<u>Examples</u>

- $\underline{G} = (G; \cdot, ^{-1}, 1)$ group. $\underline{H} \leq \underline{G}$ if $\underline{H}$ is a subgroup of $\underline{G}$
- $\underline{V} = (V; +, -, \vec{0}, (f_\cdot)_{f \in F})$ vector space.

  $\underline{W} \leq \underline{V}$ if $\underline{W}$ is a subspace of $\underline{V}$

UA $\boxed{4.3}$

? $\mathbb{R}-\mathbb{Q} \overset{?}{\leq} (\mathbb{R}, +)$ $\mathbb{R}-\mathbb{Q} \overset{?}{\leq} (\mathbb{R}, \cdot)$

? All subuniverses of $(\{a,b,c,d\}, f)$

f:

x	a	b	c	d
$f(x)$	c	c	d	d

All subalgebras of ———"———

? Subuniverses of $(\mathbb{R}^2; (*_r)_{r\in(0,1)})$

$x *_r y = r\mathbf{x} + (1-r)\mathbf{y}$

coordinate-wise

!!! $\bigcap$ subuniverses of $\underline{A}$ is a subuniverse of $\underline{A}$

$\Rightarrow$ $\mathrm{Sub}(\underline{A}) = (\text{all subuniverses of } \underline{A}, \subseteq)$ is a complete lattice

$\boxed{\text{Def}}$ $\underline{A}$ algebra, $X \subseteq A$. The subuniverse (subalgebra) generated by X is $Sg_{\underline{A}}(X)$ (or $\langle X\rangle_{\underline{A}}$) $:= \bigcap_{X\subseteq B\leq \underline{A}} B$

!!! it is the smallest (wrt. $\subseteq$) subuniverse containing X

- $X_0 := X \overset{\text{apply op's}}{\rightsquigarrow} X_1 \rightsquigarrow X_2 \rightsquigarrow$

$X_{i+1} = X_i \cup \{ f^{\underline{A}}(a_1, \dots, a_{ar(f)});\ f \in \Sigma,\ a_1, a_2, \dots \in X_i \}$

!!! $\bigcup X_i = Sg_{\underline{A}}(X) = \{ t(a_1, \dots);\ t \text{ is a "term operation" of } \underline{A} \}$

!!! $b \in Sg(X) \Rightarrow \exists Y \subseteq_{fin} X\ \ b \in Sg(Y)$ $\Rightarrow$ Sg is algebraic cl.op.

$\Downarrow$

Sub($\underline{A}$) is algebraic

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? $(\mathbb{R}^2, (*_r)_{r\in(0,1)})$ as before $x *_r y = rx + (1-r)y$

$X_0 :=$ $X_1 =$ $X_2 =$

? $X \subseteq \underline{V}$ vector space $Sg_{\underline{V}}(X) =$

? Is $(\mathbb{N}, \cdot)$ finitely generated ? what does it mean?

P PRODUCTS $\underline{A}_i = (A_i, \dots)_{i\in I}$ of the same sign. Σ

$\rightsquigarrow \prod_{i\in I} \underline{A}_i = \left(\prod_{i\in I} A_i, \dots \text{coordinate-wise} \dots\right)$

Ex $(\mathbb{Z}; +, -, 3) \times (\mathbb{R}_+; \cdot, {}^{-1}, 5) = (\mathbb{Z} \times \mathbb{R}_+; *, f, c)$ where
$(n,r) * (n',r') = (n+n', rr')$ $f((n,r)) = (-n, r^{-1})$ $c = (3,5)$

Def $\underline{A}^n = \underline{A} \times \underline{A} \times \dots \times \underline{A}$ n-th power
$\underline{A}^I = (A_i^I \dots)$ I-th power

Examples

- $(\{0,1\}, \wedge, \vee)^3 \cong (P(\{1,2,3\}), \cap, \cup)$
- vector space $\mathbb{R}^n$ is $(\mathbb{R}, +, -, 0, (r\cdot)_{r\in\mathbb{R}})^n$
- (CRT) $(\mathbb{Z}_3, +_{\text{mod } 3}) \times (\mathbb{Z}_5, +_{\text{mod } 5}) \cong (\mathbb{Z}_{15}, +_{\text{mod } 15})$

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? $(\mathbb{Z}_4, +_{\text{mod } 4}) \overset{?}{\cong} (\mathbb{Z}_2, +_{\text{mod } 2})^2$

<u>Def</u> <u>subpower</u> = subuniverse/subalgebra of a power

$R \le \underline{A}^n$ if $R \overset{\subseteq A^n}{}$ is 'compatible with operations in $\underline{A}$'

$$f: A^k \to A \text{ comp. with } R \quad \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix} \begin{matrix} \xmapsto{f} b_1 \\ \mapsto b_2 \\ \vdots \\ \mapsto b_n \end{matrix}$$

$$\begin{matrix} \cap & \cap & & \cap & & \cap \\ R & R & & R & \Longrightarrow & R \end{matrix}$$

? when $\{0\} \le \underline{A}$

? when $'\le' \le (\{0,1\}; \ldots)^2$ $\quad ("\le" = \{(a,b) \in \{0,1\}^2 : a \le b\})$

H QUOTIENTS

$\underline{A} = (A; \overset{0\text{-ary}}{f}, \overset{1\text{-ary}}{g}, \overset{2\text{-ary}}{h})$, $\sim$ equivalence on A

$\rightsquigarrow \underline{A}/\sim = (A/\sim ;$ $\underbrace{\text{naturally, i.e., by arbitrarily chosen representatives}}_{\text{must make sense}}$ $)$

<u>Recall</u>

equivalence on A:
- reflexive, symmetric, transitive relation on A ($\subseteq A^2$) eg. $\sim = \{(1,2),(2,1),(1,1),(2,2),(3,3)\}$
- partition of A eg. $\sim = |12|3|$

$A/\sim = \{a/\sim : a \in A\}$ $\quad a/\sim = \{b \in A : b \sim a\}$ e.g. ...

UA $\boxed{4.6}$

$\underline{A} = (A;\ \overset{\text{0-ary}}{f},\ \overset{\text{1-ary}}{g},\ \overset{\text{2-ary}}{h})$, $\quad \sim \leq \underline{A}^2$ equivalence relation

$\rightsquigarrow \underline{A}/_\sim = (A/_\sim;\ \underbrace{f', g', h'}_{\text{naturally}})$

when does it make sense?

$f' := f/_\sim$ (always makes sense)

$g'(a/_\sim) := g(a)/_\sim$ makes sense iff

$(a/_\sim = b/_\sim \Rightarrow g(a)/_\sim = g(b)/_\sim)$

$\Leftrightarrow (a \sim b \Rightarrow g(a) \sim g(b))$

$h'(a/_\sim, c/_\sim) := h(a,c)/_\sim$

need $(a/_\sim = b/_\sim \ \&\ c/_\sim = d/_\sim) \Rightarrow h(a,c)/_\sim = h(b,d)/_\sim$

$\Leftrightarrow (a \sim b \ \&\ c \sim d \Rightarrow h(a,c) \sim h(b,d))$

$$\begin{array}{ccccc} a & c & \xrightarrow{h} & \bullet \\ b & d & \xrightarrow{h} & \bullet \\ \wr & \wr & \Rightarrow & \wr \end{array}$$

Def. $\underline{A}$ algebra. $\sim$ is a <u>congruence</u> of $\underline{A}$

if • $\sim$ is an equivalence relation on A

and • $\sim \leq \underline{A}^2$

Then $\underline{A}/_\sim = (A/_\sim;$ naturally$)$ is a <u>quotient</u> of $\underline{A}$

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Examples

- $(S_n, \circ, {}^{-1}, id)$ $\quad \sim \; = \text{odd} \mid \text{even}$

  permutations on $\{1, \dots, n\}$

  $S_n/\sim \;\cong\; (\mathbb{Z}_2; +_{\text{mod } 2}, -_{\text{mod } 2}, 0)$

- $\underline{G}$ group, $\underline{H} \trianglelefteq \underline{G}$ def. $g \sim g'$ by $gH = g'H$

  $\underline{G}/\underline{H} = \underline{G}/\sim$

  group theory $\qquad$ UA

  in general (for groups!)
  normal subgroups
  $\updownarrow$
  congruences

- $\underline{R}$ ring, I ideal of R def. $r \sim r'$ by $r + I = r' + I$

  $\underline{R}/I = \underline{R}/\sim$

  in rings $\quad$ ideals
  $\updownarrow$
  congruences

? congruences $\leftrightarrow$ special subsets ?

NO!!!

- $\underline{A} = (A;$ no operations$)$
- lattice 

UA $\boxed{4.8}$

Examples

- $\underline{A} = (A; \dots)$, $0_A := \{(a,a);\, a \in A\} = |a_1|a_2|\dots||$

  0_A congruence $\quad \underline{A}/0_A \simeq \underline{A}$

- $\underline{A} = (A; \dots)$ $\quad 1_A := A^2 = |a_1 a_2 \dots|$

  1_A congruence $\quad \underline{A}/1_A$ one-element

trivial congruences

- $\underline{C} := \underline{A} \times \underline{B}$ $\quad \eta_{\underline{A}}, \eta_{\underline{B}}$ the projection kernels

  ($(a_1, b_1)\, \eta_{\underline{A}}\, (a_2, b_2)$ iff $a_1 = a_2$)

  η_A, η_B congruences of $\underline{C}$ $\quad \underline{A} \times \underline{B}/\eta_A \simeq \underline{A}$, ditto with η_B

Ⓞ'' $\bigcap$ congruences is a congruence

$\Rightarrow \mathrm{Con}(\underline{A}) = (\text{all congruences of } \underline{A}, \subseteq)$ is a complete lattice

Def The congruence generated by $X \subseteq A^2$ .... $Cg_{\underline{A}}(X)$

$X_0 := X \xrightarrow{\text{ref., sym., transitive cl.}} Y_0 \xrightarrow{\text{apply op's}} X_1 \xrightarrow{\text{trans}} Y_1 \xrightarrow{\text{apply}} \dots\dots$

the same Ⓞ''' as for $\mathbf{S}$ $\Rightarrow Cg$ is an algebraic cl.op on A^2

$\Rightarrow \mathrm{Con}(\underline{A})$ is algebraic

Ⓞ'''' $\mathrm{Con}(\underline{A}) \leq_{\text{complete}} \mathrm{Eq}(A)$: joins are computed as in Eq !