Universal Algebra 1 – Exercises 1

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29 September 2022, Prague

Exercise 1.

(i) Let $\mathbf{S} = \langle S, \wedge \rangle$ be a semilattice. Find and define a partial order relation $\leq$ on S such that $\langle S, \leq \rangle$ is a semilattice ordered set.

(ii) Let $\langle S, \leq \rangle$ be a semilattice ordered set. Find and define a binary operation $\wedge$ on S such that $\langle S, \wedge \rangle$ is a semilattice.

(iii) Verify that the composition of the constructions (i) and (ii) acts as the identity on the semilattice $\mathbf{S} = \langle S, \wedge \rangle$.

(iv) Verify that the composition of the constructions (ii) and (i) acts as the identity on the semilattice ordered set $\langle S, \leq \rangle$.

(v) Conclude that the definitions of semilattice and semilattice ordered set are equivalent.

Exercise 2.

(i) Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice. Find and define a partial order relation $\leq$ on L such that $\langle L, \leq \rangle$ is a lattice ordered set.

(ii) Let $\langle L, \leq \rangle$ be a lattice ordered set. Find and define two binary operations $\wedge$ and $\vee$ on L such that $\langle L, \wedge, \vee \rangle$ is a lattice.

(iii) Verify that the composition of the constructions (i) and (ii) acts as the identity on the lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$.

(iv) Verify that the composition of the constructions (ii) and (i) acts as the identity on the lattice ordered set $\langle L, \leq \rangle$.

(v) Conclude that the definitions of lattice and lattice ordered set are equivalent.

Exercise 3. Are there isomorphic algebras among the following families?

(i) $\langle \mathbb{Q}, +, 0 \rangle,\ \langle \mathbb{R}, +, 0 \rangle$

(ii) $\langle \mathbb{N}, \cdot \rangle,\ \langle 2\mathbb{N}, \cdot \rangle,\ \langle 3\mathbb{N}, \cdot \rangle,\ \langle 2\mathbb{N}+1, \cdot \rangle$

(iii) $\langle \mathbb{C}, + \rangle,\ \langle \mathbb{R}, + \rangle \times \langle \mathbb{R}, + \rangle$

(iv) $\langle \mathbb{C}, \cdot \rangle,\ \langle \mathbb{R}, \cdot \rangle \times \langle \mathbb{R}, \cdot \rangle$

Exercise 4. Which of the following Hasse diagrams represent a lattice?