Universal Algebra 1 – Exercises 2

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Exercise 1. Prove that every distributive lattice is modular.

Exercise 2. Prove that every lattice of order $n \leq 4$ is distributive.

Exercise 3. Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice. Prove that $\mathbf{L}$ is distributive if and only if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for every $x, y, z \in L$.

Exercise 4. Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice. Prove that for every $x, y, z \in L$

(i) $x \vee (y \wedge z) \geq (x \vee y) \wedge (x \vee z)$.

(ii) $x \leq y \implies x \wedge (y \vee z) \leq x \vee (y \wedge z)$.

Conclude that to prove whether a lattice is distributive/modular or not it is enough to verify the reverse inequalities.

Exercise 5.

(i) Is distributivity of lattices invariant under isomorphism?

(ii) Is modularity of lattices invariant under isomorphism?

Exercise 6. Construct two lattices $\mathbf{M}$ and $\mathbf{N}$ with the two following properties

(i) For every lattice $\mathbf{L}$, if $\mathbf{L}$ contains a copy of $\mathbf{N}$ then $\mathbf{L}$ is not modular.

(ii) For every lattice $\mathbf{L}$, if $\mathbf{L}$ contains either a copy of $\mathbf{N}$ or a copy of $\mathbf{M}$ then $\mathbf{L}$ is not distributive.

Exercise 7. Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a distributive lattice. Define the *majority terms*

$$m_1(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

$$m_2(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z).$$

Prove that $m_1(x, y, z) = m_2(x, y, z)$ for every $x, y, z \in L$.

Exercise 8. Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a modular lattice, and let $a, b \in L$. Prove that the intervals $\mathbf{I}[a \wedge b, a]$ and $\mathbf{I}[b, a \vee b]$ are isomorphic (as sublattices of $\mathbf{L}$).

Exercise 9. Let $\mathbf{G} = \langle G, \cdot, ^{-1}, 1\rangle$ be a group. Denote by $\mathrm{Sub}(\mathbf{G})$ the family of subgroups and by $\mathrm{Nml}(\mathbf{G})$ the family of normal subgroups of $\mathbf{G}$), respectively.

(i) Prove that $\langle \mathrm{Sub}(\mathbf{G}), \subseteq\rangle$ is a lattice ordered set.

(ii) Prove that $\langle \mathrm{Nml}(\mathbf{G}), \subseteq\rangle$ is a lattice ordered set.

(iii) Prove or disprove: $\mathrm{Sub}(\mathbf{G})$ is modular.

(iv) Prove or disprove: $\mathrm{Sub}(\mathbf{G})$ is distributive.

(v) Prove or disprove: $\mathrm{Nml}(\mathbf{G})$ is modular.

(vi) Prove or disprove: $\mathrm{Nml}(\mathbf{G})$ is distributive.

Exercise 10. Draw the Hasse diagram of the lattice $\langle \mathrm{Eq}(X)), \subseteq\rangle$, where $X = \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}$. For which X is it distributive or modular?

Exercise 11. Let X be a set. Find and prove formulas to compute meet and join in the lattice $\langle \mathrm{Eq}(X), \subseteq\rangle$.