Universal Algebra 1 – Exercises 4

Filippo Spaggiari

20 October 2022, Prague

Exercise 1. Let $\mathbf{A} = \langle A, \mathcal{F} \rangle$ be an algebra. Is $\emptyset$ a subuniverse of $\mathbf{A}$? Is $\emptyset$ a subalgebra of $\mathbf{A}$?

Exercise 2. Let $\mathbf{A} = \langle A, * \rangle$ be a binary algebra and let $\theta \in \mathrm{Eq}(A)$. Prove that $\theta \in \mathrm{Con}(\mathbf{A})$ if and only if for every $a, b \in A$ we have

$$(a,b) \in \theta \implies \begin{cases} (a*c, b*c) \in \theta \\ (c*a, c*b) \in \theta \end{cases} \quad \forall c \in A$$

Exercise 3. Let $\mathbf{A} = \langle A, * \rangle$ be the algebra where $A = \{0, 1, 2, 3\}$ and $*$ is the binary operation defined by the following multiplication table

$*$	0	1	2	3
0	0	2	1	1
1	2	1	0	2
2	1	0	2	0
3	1	2	0	3

Determine all the subuniverses, the subalgebras, and the congruences of $\mathbf{A}$, then draw the lattices $\mathrm{Sub}(\mathbf{A})$ and $\mathrm{Con}(\mathbf{A})$.

Exercise 4. Prove that the algebra $\langle \mathsf{M}_2(\mathbb{Z}), +, -, \cdot \rangle$ is generated by the set $\left\{ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \right\}$.

Exercise 5. Let $\mathbf{A}$ be an algebra and let $\theta \in \mathrm{Eq}(A)$. Prove that $\theta \in \mathrm{Con}(\mathbf{A})$ if and only if θ is a subalgebra of $\mathbf{A}^2$.

Exercise 6. Let $\mathbf{A}$ and $\mathbf{B}$ be algebras and let $h \colon \mathbf{A} \to \mathbf{B}$ be a homomorphism.

(i) Let $U \in \mathrm{Sub}(\mathbf{A})$ and $V \in \mathrm{Sub}(\mathbf{B})$. Are necessarily $\overrightarrow{h}(U) \in \mathrm{Sub}(\mathbf{B})$ and/or $\overleftarrow{h}(V) \in \mathrm{Sub}(\mathbf{A})$?

(ii) Let $\theta \in \mathrm{Con}(\mathbf{A})$ and $\psi \in \mathrm{Con}(\mathbf{B})$. Are necessarily $\overrightarrow{h}(\theta) \in \mathrm{Con}(\mathbf{B})$ and/or $\overleftarrow{h}(\psi) \in \mathrm{Con}(\mathbf{A})$?

(iii) Let $X \subseteq A$. Is it true that $\overrightarrow{h}(\mathrm{Sg}^{\mathbf{A}}(X)) = \mathrm{Sg}^{\mathbf{B}}(\overrightarrow{h}(X))$?

Exercise 7. Let $\mathbf{2} = \langle \{0, 1\}, \wedge, \vee \rangle$ be the two element lattice. Prove that for every set X there is a lattice isomorphism $\langle \mathcal{P}(X), \cap, \cup \rangle \cong \mathbf{2}^X$.