Universal Algebra 1 – Exercises 5

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Exercise 1. Prove that the composition of homomorphisms is a homomorphism.

Exercise 2. Prove that the inverse of an isomorphism is an isomorphism.

Exercise 3. Let $h\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism. Prove that h is injective if and only if $\ker(h) = 0_{\mathbf{A}}$.

Exercise 4. Let $\mathbf{A}$ and $\mathbf{B}$ be similar algebras, and let $h\colon A \to B$ be a function. Prove that h is a homomorphism if and only if $h \in \mathrm{Sub}(\mathbf{A} \times \mathbf{B})$.

Exercise 5. Let $g, h\colon \mathbf{A} \to \mathbf{B}$ be homomorphisms, and let $X \subseteq A$ be a generating set of $\mathbf{A}$. Prove that if $g{\restriction_X} = h{\restriction_X}$ then $g = h$.

Exercise 6. Let $f\colon \mathbf{A} \to \mathbf{B}$ and $g\colon \mathbf{A} \to \mathbf{C}$ be homomorphisms, with g surjective. Prove that if $\ker(g) \subseteq \ker(f)$ then there is a homomorphism $h\colon \mathbf{C} \to \mathbf{B}$ such that $f = h \circ g$. Draw an appropriate commuting diagram.

Exercise 7. Let $h\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism, and let $\psi \in \mathrm{Con}(\mathbf{B})$. Prove that there is an embedding $h/\psi\colon \mathbf{A}/\overleftarrow{h}(\psi) \to \mathbf{B}/\psi$. Draw an appropriate commuting diagram.

Exercise 8. Find all homomorphisms from $\langle \mathbb{N}, + \rangle \times \langle \mathbb{N}, + \rangle$ to $\langle \{+1, -1\}, \cdot \rangle$.

Exercise 9. Let $\mathbf{A}$ and $\mathbf{B}$ be similar algebras and let X be a generating set of $\mathbf{A}$. Prove or disprove the following statements.

(i) Every arbitrarily defined function $h\colon X \to B$ extends to a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$.

(ii) Assume that X is a *minimal generating set* of $\mathbf{A}$, that is, every proper subset of X does not generate $\mathbf{A}$. Then, every arbitrarily defined function $h\colon X \to B$ extends to a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$.

Exercise 10. Find a class of algebras $\mathcal{K}$ and a finite class of algebras $\mathcal{H}$ which are not closed under the operator H. Repeat with the operators S and P.

Exercise 11.* Prove that the class operator inclusions $\mathsf{SH} \leq \mathsf{HS}$, $\mathsf{PS} \leq \mathsf{SP}$, $\mathsf{PH} \leq \mathsf{HP}$ may be strict.

Exercise 12.* Prove that every finitely generated variety is locally finite.